Sample final Solutions

Q1 a) $$f'(x) = \frac{(x-1)\frac{1}{x^2+x}(2x+1) - \ln(x^2+x).(1)}{(x-1)^2}.$$

b) $$f'(x) = -3e^{-3x}\tan(x+1) + e^{-3x}\sec^2(x+1).1.$$

c) $$f'(x) = \frac{1}{3}\left(e^{\sin(x)}+1\right)^{-2/3}\left(e^{\sin(x)}.\cos x\right)$$

Q2 a) $$\int \frac{x^2+2x+1}{x^{1/3}}\,dx = \int x^{5/3} + 2x^{2/3} + x^{-1/3}\,dx$$

$$= \frac{3x^{8/3}}{8} + \frac{6}{5}x^{5/3} + \frac{3x^{2/3}}{2} + C$$

b) $$\int_0^{\pi/6} \sin^2(3x)\cos(3x)\,dx \qquad u = \sin(3x)$$

$$\frac{du}{dx} = \cos(3x).3.$$

$$\int_0^{\sqrt{3}/2} u^2\cos(3x)\frac{dx}{du}\,du = \int_0^{\sqrt{3}/2} u^2\cos(3x)\frac{1}{3\cos(3x)}.du$$

$$= \int_0^{\sqrt{3}/2}\frac{1}{3}u^2\,du = \left[\frac{1}{9}u^3\right]_0^{\sqrt{3}/2} = \frac{1}{9}\left(\frac{\sqrt{3}}{2}\right)^3 = \frac{\sqrt{3}}{8}.$$

c) $$\int \frac{1}{9+x^2}\,dx = \frac{1}{9}\int \frac{1}{1+(x/3)^2}\,dx \qquad \text{sub } u = \frac{x}{3} \quad \frac{du}{dx} = \frac{1}{3}.$$

$$\frac{1}{9}\int \frac{1}{1+u^2}\frac{dx}{du}\,du = \frac{1}{3}\int\frac{1}{1+u^2}\,du = \frac{1}{3}\tan^{-1}(u) + C = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right).$$

Q3 a) $\lim_{x\to 3} \frac{x^2 - x - 6}{x-3} = \lim_{x\to 3} \frac{2x-1}{1} = 5.$

b) $\lim_{x\to 0} \frac{\sin 5x}{e^x - 1} = \lim_{x\to 0} \frac{\cos(5x) \cdot 5}{e^x} = 5.$

c) $\lim_{x\to 0+} e^{\ln(x)\sin x} = e^{\lim_{x\to 0+} \ln(x)\sin(x)}$

$\lim_{x\to 0+} \frac{\ln(x)}{1/\sin(x)} = \lim_{x\to 0+} \frac{\frac{1}{x}}{-\sin(x)^{-2}\cdot\cos(x)} = \lim_{x\to 0+} \frac{-\sin^2 x}{x\cos x}.$

$= \lim_{x\to 0+} \frac{-2\sin x \cdot \cos x}{\cos x + x \cdot -\sin x} = \frac{0}{1} = 0$

so $\lim_{x\to 0+} e^{\ln(x)\sin x} = e^0 = 1.$

d) $\lim_{x\to 0} x\sin(\frac{1}{x})$ squeeze thm $-|x| \leq x\sin(\frac{1}{x}) \leq |x|$

$\lim_{x\to 0} -|x| = 0$ $\lim_{x\to 0} |x| = 0$

$\Rightarrow \lim_{x\to 0} x\sin(\frac{1}{x}) = 0.$

Q4 $f(x) = x^3 - 3x$

a) $f'(x) = 3x^2 - 3$ critical points solve $f'(x) = 0$ $x^2 = 1$
$x = \pm 1.$

b) $3x^2 - 3 = 3(x+1)(x-1)$

$f'(x)$: $+$ | $-$ | $+$
-1 $+1$

increasing $\Leftrightarrow f'(x) > 0$ $(-\infty,-1)\cup(1,\infty)$
decreasing $\Leftrightarrow f'(x) < 0$ $(-1,1).$

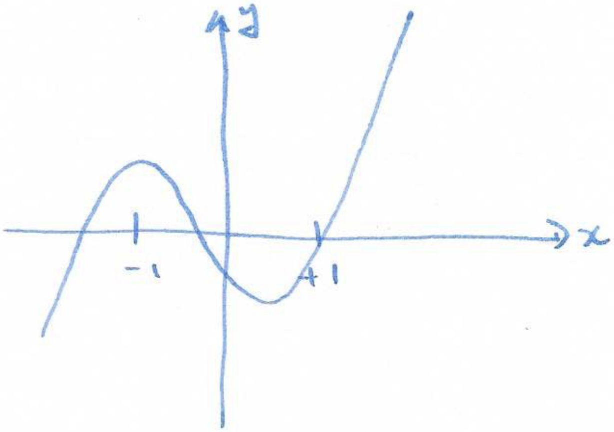
c) $f''(x) = 6x$ concave up $\leftrightarrow f''(x) > 0 \leftrightarrow x > 0$ $(0, \infty)$

concave down $\leftrightarrow f''(x) < 0$ $(-\infty, 0)$.

d) $+1$ concave up min

-1 concave down max.

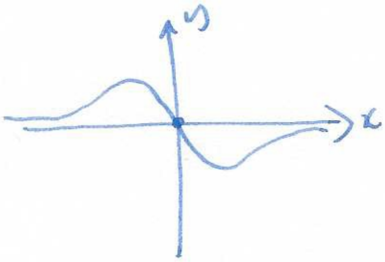
e)



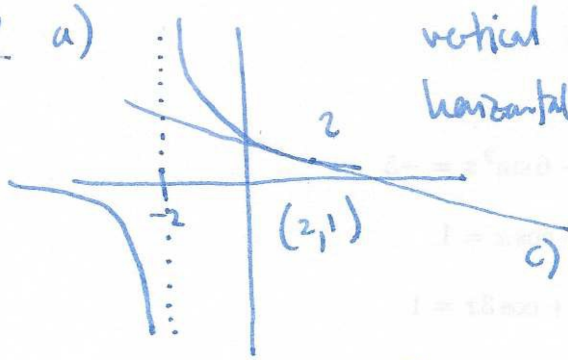
Q5 a) $(-5, -\frac{3}{2}) \cup (\frac{3}{2}, 5)$

b) $(-5, 0)$.

c)



Q6 a)



vertical asymptote $x = 2$

horizontal asymptote $y = 0$

b) $f'(x) = -4(x+2)^{-2} \cdot 1$ $\qquad f'(2) = \dfrac{-4}{4^2} = -\dfrac{1}{4}$.

$y - 1 = -\frac{1}{4}(x - 2)$

Q7 $f(x) = x^2 - 2x$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - 2(x+h) - x^2 + 2x}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} + 2xh + h^2 - \cancel{2x} - 2h - \cancel{x^2} + \cancel{2x}}{h} = \lim_{h\to 0} 2x + h - 2 = 2x - 2.$$

Q8 $xy^3 + 2x^2y^2 - x^2y = 10$

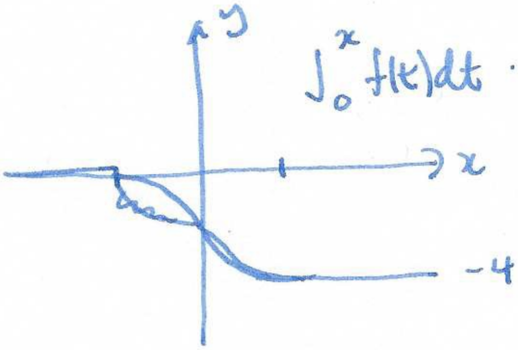
$$y^3 + x\,3y^2\frac{dy}{dx} + 4xy^2 + 2x^2\,2y\frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} = 0$$

at $(2,-1)$: $\quad -1 + 6y' + 8 + -16y' - 4 - y' = 0$

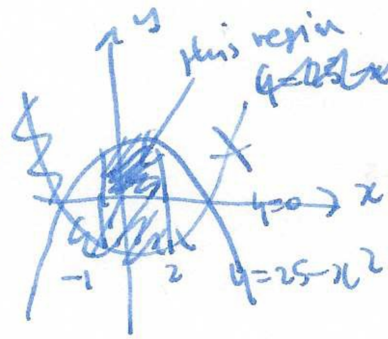
$$-11y' + 3 = 0 \qquad y' = 3/11.$$

$$y + 1 = \frac{3}{11}(x-2).$$

Q9



Q10 a)



b) $$\int_{-1}^{2} 25 - x^2\,dx = \left[25x - \frac{1}{3}x^3\right]_{-1}^{2}$$

$$= 50 - \frac{8}{3} + 25 - \frac{1}{3} = 72.$$

Q11 $A = \pi r^2 \qquad \underbrace{\frac{dA}{dt}}_{5} = 2\pi \underbrace{r}_{12} \frac{dr}{dt} \qquad \frac{dr}{dt} = \frac{5}{24\pi}$ m/s.

Q12

$f(x) = \sqrt[3]{x} = x^{1/3}$.

$f'(x) = \frac{1}{3}x^{-2/3}$.

$f(x+\Delta x) \approx \cancel{f'(x)}\, f(x) + f'(x)\Delta x$

$x = \cancel{81}\ 64$

$\Delta x = \cancel{-1}\ 16$

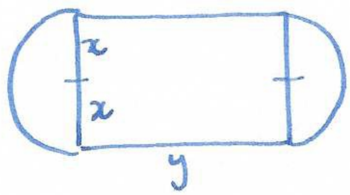
$f(80) \approx (64)^{1/3} + \frac{1}{3}(64)^{-2/3}\cdot 16 = 4 + \frac{1}{3.16}\cdot 16 = 4 + \frac{1}{3}$.

exact value : $\sqrt[3]{80} = 4.309$

absolute error $= 0.024$

percentage error $= \frac{0.024}{4.309} \times 100 \approx 0.58\%$

Q13



$A = 2xy + \pi x^2$

$P = 2y + 2\pi x = 400$

$y = 200 - \pi x$

$A = 2x(200 - \pi x) + \pi x^2 = 400x - \pi x^2$

$\frac{dA}{dx} = 400 - 2\pi x$ critical point solve $\frac{dA}{dx} = 0$: $x = \frac{200}{\pi}$ $y = 0$.