§5.2 Diagonalization

Example $A = \begin{bmatrix} 5 & 3 \\ -6 & -4 \end{bmatrix}$ has eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalue 2

$\begin{bmatrix} -1 \\ 2 \end{bmatrix}$ $\qquad -1$

$$\begin{tikzcd} \underset{\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}{\mathbb{R}^2} \arrow[r, "A"] & \underset{\left\{\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\1\end{bmatrix}\right\}}{\mathbb{R}^2} \\ \underset{\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}\right\}}{\mathbb{R}^2} \arrow[u, "S"] \arrow[r, "\begin{bmatrix}2&0\\0&-1\end{bmatrix}"] & \underset{\left\{\begin{bmatrix}1\\-1\end{bmatrix},\begin{bmatrix}-1\\2\end{bmatrix}\right\}}{\mathbb{R}^2} \arrow[u, "S"'] \end{tikzcd}$$

Q: what is matrix for A wrt to basis $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$?

A: $A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = -1\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

so $A_{\substack{\text{new}\\\text{basis}}} = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$.

Let S be the matrix that takes a vector in the $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$ basis to a vector in the $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ basis, i.e. $S = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$

S is the <u>matrix of eigenvectors.</u> (if there are enough eigenvectors!).

note $\quad S^{-1}AS = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$ $\quad$ (check!)

In general: suppose $\underset{n\times n}{A}$ has n linearly independent eigenvectors $v_1, \ldots, v_n$ with eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $S = \begin{bmatrix} | & | & & | \\ v_1 & v_2 & \cdots & v_n \\ | & | & & | \end{bmatrix}$

then

$$\boxed{S^{-1}AS = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}}$$

equivalently

$$A = S\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}S^{-1}$$

Remarks

① if all eigenvalues $\lambda_1, \ldots, \lambda_n$ are distinct, then their an n independent eigenvectors.

② S is not unique

③ not all matrices have n independent eigenvalues.

problem: repeated eigenvalues.

examples $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ $\lambda_1=\lambda_2=0.$ $\det(A-\lambda I)=\begin{vmatrix}-\lambda&1\\0&-\lambda\end{vmatrix}=\lambda^2=0.$

but $\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x_2=0$ so $x=\begin{bmatrix}t\\0\end{bmatrix}=t\begin{bmatrix}1\\0\end{bmatrix}$

only 1 eigenvector.

Defⁿ the algebraic multiplicity of an eigenvalue is the power of $(\lambda-\lambda_i)$ in $\det(A-\lambda I)=0.$

the geometric multiplicity is $\dim(N(A-\lambda_i I))$

Note geometric multiplicity $\leq$ algebraic multiplicity

$<$ $\Rightarrow$ not diagonalizable.

Proof $\begin{bmatrix}0&1\\0&0\end{bmatrix}$ not diagonalizable. $D=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ but $S^{-1}\begin{bmatrix}0&1\\0&0\end{bmatrix}S=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}0&1\\0&0\end{bmatrix}=S\begin{bmatrix}0&0\\0&0\end{bmatrix}S^{-1}$

$=\begin{bmatrix}0&0\\0&0\end{bmatrix}$ $\#$ $\square$.

Example. $\begin{bmatrix}1&1\\0&1\end{bmatrix}$ also not diagonalizable.

$\lambda_1=\lambda_2=1$ $v_1=\begin{bmatrix}1\\0\end{bmatrix}$ $S^{-1}\begin{bmatrix}1&1\\0&1\end{bmatrix}S=\begin{bmatrix}1&0\\0&1\end{bmatrix}.$

$\begin{bmatrix}1&1\\0&1\end{bmatrix}=S^{-1}\begin{bmatrix}1&0\\0&1\end{bmatrix}S=\begin{bmatrix}1&0\\0&1\end{bmatrix}$ $\#$.

Lemma (5D) Eigenvectors corresponding to distinct eigenvalues are independent

Proof Let v_1 have eigenvalue λ_1 $Av_1=\lambda v_1$

v_2 λ_2 $Av_2=\lambda v_2$

spose $c_1v_1+c_2v_2=0$ ①

then $A(c_1v_1+c_2v_2)=c_1Av_1+c_2Av_2=c_1\lambda_1v_1+c_2\lambda_2v_2=0$ ②

$\lambda_1 ① - ②$: $\left.\begin{array}{l} c_1\lambda_1 v_1 - c_2\lambda_1 v_2 = 0 \\ c_1\lambda_1 v_1 - c_2\lambda_2 v_2 = 0 \end{array}\right\}$ $c_2(\lambda_2-\lambda_1)v_2 = 0$

$\Rightarrow c_2 = 0.$

$① - \lambda_2 ①$: $\Rightarrow c_1 = 0.$ □.

Example projection matrix $A = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ eigenvalues : $(1/2-\lambda)^2 - \frac{1}{4} = 0$

$\lambda^2 - \lambda = 0$

$\lambda(\lambda-1) = 0$

$\lambda = 0, 1.$

eigenvalues $\begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Example rotation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\det(A-\lambda I) = (-\lambda)^2 + 1 = \lambda^2 + 1 = 0$ $\lambda = \pm i$

eigenvectors $\begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}$ check $S^{-1}AS = D$!

Powers A^k : if v_i is an eigen$\overset{\text{ector}}{\text{value}}$ of A with eigenvalue λ_i

then v_i A^k λ_i^k.

check: $A^k v_i = A \cdots \underbrace{A v_i}_{\lambda v_i} = \underbrace{A \cdots A}_{k-1} \lambda_i v_i = \cdots \lambda_i^k v_i$

so if $S^{-1}AS = D$ then $S^{-1}A^kS = D^k$

alternate proof: $(S^{-1}AS)^k = S^{-1}A\underbrace{S\,S^{-1}}_{I}A\underbrace{S}_{I}\cdots\underbrace{S^{-1}}_{I}AS = D^k.$

$= S^{-1}A^kS$

also: eigenvalues of A^{-1} are $\frac{1}{\lambda_i}$ proof: $A v_i = \lambda_i v_i$

$v_i = \lambda_i A^{-1} v_i$

$\frac{1}{\lambda_i} v_i = A^{-1} v_i.$ □.

§5.3 Applications

Difference equations interest. P_k = amount of money

$P_{k+1} = (1.05) P_k$.

solution: $P_k = (1.05)^k P_0$

Fibonacci numbers 1, 1, 2, 3, 5, 8, ... $F_{n+2} = F_{n+1} + F_n$

set $u_k = \begin{bmatrix} F_{k+1} \\ F_k \end{bmatrix}$ then $u_{k+1} = \begin{bmatrix} F_{k+2} \\ F_{k+1} \end{bmatrix} = \begin{bmatrix} F_{k+1} + F_k \\ F_{k+1} \end{bmatrix}$.

so $u_{k+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} u_k$ $u_{k+1} = A u_k$ $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

closed form: $u_k = A^k u_0$

recall: if $A = SDS^{-1}$ then $A^k = SD^kS^{-1}$

eigenvalues: $(1-\lambda)(-\lambda) - 1 = 0$ $\lambda^2 - \lambda - 1 = 0$

$\lambda = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$ $\left(\underline{u_k} \Rightarrow F_k = a\left(\frac{1-\sqrt5}{2}\right)^k + b\left(\frac{1+\sqrt5}{2}\right)^k \right)$

eigenvectors:

$\lambda = \frac{1-\sqrt5}{2}$: $\begin{bmatrix} \frac12 + \frac{\sqrt5}{2} & 1 \\ 1 & -\frac12 + \frac{\sqrt5}{2} \end{bmatrix} \begin{bmatrix} \frac12 - \frac{\sqrt5}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac14 - \frac54 + 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\lambda = \frac{1+\sqrt5}{2}$: $\begin{bmatrix} \frac12 - \frac{\sqrt5}{2} & 1 \\ 1 & -\frac12 - \frac{\sqrt5}{2} \end{bmatrix} \begin{bmatrix} +\frac12 + \frac{\sqrt5}{2} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac14 - \frac54 + 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$D = \begin{bmatrix} \frac{1-\sqrt5}{2} & 0 \\ 0 & \frac{1+\sqrt5}{2} \end{bmatrix}$ $S^{-1} = \begin{bmatrix} \frac12 - \frac{\sqrt5}{2} & \frac12 + \frac{\sqrt5}{2} \\ 1 & 1 \end{bmatrix}$.

Markov chains

Example

each year $\frac{1}{10}$ of people outside CA move to CA
$\frac{2}{10}$ in CA move out of CA.

$$\begin{bmatrix} y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0.9 & 0.2 \\ 0.1 & 0.8 \end{bmatrix} \begin{bmatrix} y_0 \\ z_0 \end{bmatrix} \qquad x_k = A^k x_0 .$$

↑ properties: each column adds up to 1
non-negative entries.

find eigenvalues: $\det(A - \lambda I) = \lambda^2 - 1.7\lambda\ 0.7 = (\lambda - 1)(\lambda - 0.7)$

$$SDS^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.7 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{bmatrix} y_k \\ z_k \end{bmatrix} = SD^kS^{-1} = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 1^k & 0 \\ 0 & 0.7^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} = (y_0 + z_0) \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} + (y_0 - 2z_0)(0.7)^k \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} (y_0 + z_0) \quad \text{as } k \to \infty .$$

Population example

stability of $A^k x_0 = SD^kS^{-1}x_0 = c_1 \lambda_1^k x_1 + \dots + c_n \lambda_n^k x_n$

stable all eigenvalues $|\lambda_i| < 1$ $(\Rightarrow A^k \to 0)$.

neutrally stable if some $|\lambda_i| = 1$ and all other $|\lambda_i| < 1$

unstable if at least one $|\lambda_i| > 1$

recall if $A = SDS^{-1}$ $\quad D = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

then $A^k = (SDS^{-1})(SDS^{-1})\cdots(SDS^{-1}) = SD^kS^{-1}$

$$D^k = \begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}$$

Q: what should e^{At} be? $\quad e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots$

$$e^{At} = 1 + At + A^2\frac{t^2}{2!} + A^3\frac{t^3}{3!} + \cdots$$

$$e^{SDS^{-1}} = 1 + SDS^{-1}t + SD^2S^{-1}\frac{t^2}{2!} + SD^3S^{-1}\frac{t^3}{3!} + \cdots$$

$$= S\left(1 + Dt + D^2\frac{t^2}{2!} + \cdots\right)S^{-1} = S\begin{pmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{pmatrix}S^{-1}$$

not every matrix can be diagonalized

Jordan normal form

Def^n A Jordan block is a matrix of the form $\begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}$. (1x1 case $[\lambda]$)

Th^m Every matrix A has a matrix S s.t. $A = SJS^{-1}$

where $J = \begin{bmatrix} \boxed{\lambda_1} & & & \\ & \boxed{\lambda_2} & & \\ & & \boxed{\lambda_3} & \\ & & & \boxed{\lambda_4} \end{bmatrix}$ Jordan blocks on the diagonal