How to find eigenvectors and eigenvalues.

$$Av = \lambda v = \lambda I v$$

$$Av - \lambda I v = 0$$

$\underbrace{(A-\lambda I)}_{n \times n \text{ matrix}} v = 0$ v lives in the nullspace / kernel for $A-\lambda I$

recall: $N(A-\lambda I) > 0 \iff \det(A-\lambda I) = 0$

Def^n $\det(A-\lambda I)=0$ is called the **characteristic equation**

to find eigenvalues, solve $\det(A-\lambda I)$.

Example $A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} = (2-\lambda)(1-\lambda) = 0.$

$\lambda = 2, 1.$

$A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ $\det(A-\lambda I) = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) + 2$

$= \lambda^2 - 3\lambda + 2 = (\lambda-2)(\lambda-1) = 0$

$\lambda = 2, 1.$

note 2×2: $\det(A-\lambda I)$ is quadratic 2 solution
3×3: $\det(A-\lambda I)$ is cubic 3 solution
$n\times n$: $\det(A-\lambda I)$ is degree n. n solutions
} counting (with multiplicity).

Once you've found the eigenvalues, you can find the eigenvectors.

Example $\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ $\underline{\lambda = 1}$ solve $(A-I)x = 0$

$$\begin{bmatrix} 3-1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} \rightsquigarrow \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix} \end{array}$$

t

$2x_1 - t = 0$

$x_1 = t/2.$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} = t' \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$\underline{\lambda=2}$: solve $(A-2I)x = 0$ $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ $x_1 - t = 0$

t

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

summary $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$ has eigenvector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with eigenvalue 2

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ 1.

Observation · if v is an eigenvalue, so is any multiple of v.

· dim $(A-\lambda I)$ may be >1.

Example. $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ eigenvalue $\lambda = 1$ has eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

2 $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

Observation : special case of triangular matrices.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 0 & 4-\lambda & 5 \\ 0 & 0 & 6-\lambda \end{bmatrix}$ $\det(A-\lambda I) = (1-\lambda)(4-\lambda)(6-\lambda)$

so eigenvalues are diagonal elements.

Useful facts · $\det(A)$ = product of eigenvalues $\lambda_1 \dots \lambda_n$

· $\text{trace}(A) = a_{11} + a_{22} + \dots + a_{nn} = \lambda_1 + \lambda_2 + \dots + \lambda_n$ sum of eigenvalues.

Warning : row operations do not preserve eigenvalues! $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. different!

Examples $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1$

$$\lambda = \frac{3 \pm \sqrt{9-4}}{2} = \frac{3 \pm \sqrt{5}}{2}.$$

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\begin{bmatrix} -\lambda & -1 \\ 1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0$ $\lambda = \pm i$.

find eigenvectors : $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} -i & -1 \\ 0 & \underset{t}{0} \end{bmatrix}$ $v = t \begin{bmatrix} 1 \\ -i \end{bmatrix}$ check $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -i \end{bmatrix} = \begin{bmatrix} i \\ 1 \end{bmatrix} = i \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

§5.2 Diagonalization