§4.4 Applications of determinants

① Find A^{-1}: a formula for A^{-1}

Example (2x2) $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det A}\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}^T = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

C_{ij} = ij-cofactor (remember sign! $(-1)^{i+j}$)

in general ($n \times n$) $A^{-1} = \frac{1}{\det A} C^T$ $\qquad C^T$ = matrix of cofactors.

check: $$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}\begin{bmatrix} C_{11} & \cdots & C_{n1} \\ \vdots & & \\ C_{1n} & \cdots & C_{nn} \end{bmatrix} = \begin{bmatrix} \det(A) & & 0 \\ & \ddots & \\ 0 & & \det(A) \end{bmatrix}$$

1st row, first col: $a_{11}C_{11} + a_{12}C_{21} + a_{13}C_{31} + \cdots a_{1n}C_{n1} = \det A$.

off diagonals are zero: 1st row, 2nd col: $a_{11}C_{21} + a_{12}C_{22} + \cdots + a_{1n}C_{2n}$

= det of a matrix with two rows the same! $\begin{bmatrix} a_{11} \cdots a_{1n} \\ a_{11} \cdots a_{1n} \\ a_{31} \cdots a_{3n} \\ \end{bmatrix} = 0$.

Q: why do we want a formula?

A: shows: entries of A^{-1} are degree n rational functions in entries of A

— det(A), A^{-1} depends continuously on A.

② solve $Ax = b$ (if A $n \times n$ full rank)

$$x = A^{-1}b = \frac{1}{\det A} C^T b$$

Cramer's rule $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\qquad x_j = \frac{\det B_j}{\det A}$ $\qquad$ b in j-th col.

$$B_j = \begin{bmatrix} a_{11} & a_{12} & b_1 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & b_n & a_{nn} \end{bmatrix}$$

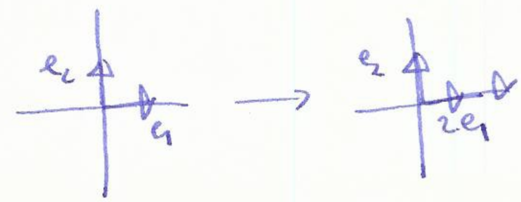
Proof expand out along row j □

__note__: solution vary continuously with coeffs in A.

problem: if $\det(A)$ very close to zero small errors in b get magnified.

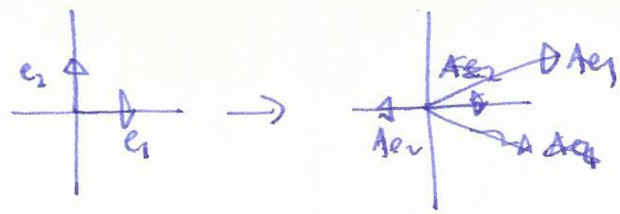
§5.1 Eigenvalues and eigenvectors

__Example__ $A:\mathbb{R}^2\to\mathbb{R}^2$ $\quad A=\begin{bmatrix}2&0\\0&1\end{bmatrix}$



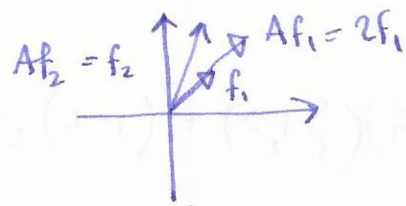
what about: $\cancel{\begin{bmatrix}3&1\\-2&0\end{bmatrix}}$

$\begin{bmatrix}3&-1\\2&0\end{bmatrix}$



__note__: $\begin{bmatrix}3&-1\\2&0\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}$

$\begin{bmatrix}3&-1\\2&0\end{bmatrix}\begin{bmatrix}1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$



__observation__ $\left\{\begin{bmatrix}1\\1\end{bmatrix},\begin{bmatrix}1\\2\end{bmatrix}\right\}$ is a __basis__ for $\mathbb{R}^2$, so we can write any vector

$$x = c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\2\end{bmatrix} \quad \text{then } Ax = c_1A\begin{bmatrix}1\\1\end{bmatrix}+c_2A\begin{bmatrix}1\\2\end{bmatrix} = 2c_1\begin{bmatrix}1\\1\end{bmatrix}+c_2\begin{bmatrix}1\\2\end{bmatrix}$$

$$Ax = c_1f_1+c_2f_2 = 2c_1f_1+c_2f_2.$$

$$A\begin{bmatrix}c_1\\c_2\end{bmatrix}_f=\begin{bmatrix}2c_1\\c_2\end{bmatrix}_f \quad \text{i.e. } A_f=\begin{bmatrix}2&0\\0&1\end{bmatrix}.$$

__Def^n__ An __eigenvector__ v for a A is a vector v s.t. $Av=\lambda v$. λ is called the __eigenvalue__ for v. ($v\neq 0$ zero vector, but $\lambda=0$ is ok)

__Example__.

$\begin{bmatrix}2&0\\0&1\end{bmatrix}$ has eigenvectors	$\begin{bmatrix}1\\0\end{bmatrix}$	with eigenvalues	2
	$\begin{bmatrix}0\\1\end{bmatrix}$		1
$\begin{bmatrix}3&-1\\2&0\end{bmatrix}$	$\begin{bmatrix}1\\1\end{bmatrix}$		2
	$\begin{bmatrix}1\\2\end{bmatrix}$		1