note can expand out along any row or column.

Example

$$\det \begin{bmatrix} 0 & -1 & 1 & 2 \\ -1 & 1 & 0 & 1 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 4 & 1 \end{bmatrix} = 3 \begin{vmatrix} 0 & -1 & 2 \\ -1 & 1 & 1 \\ 1 & 2 & 1 \end{vmatrix}$$

Permutations note 3x3 case:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{vmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{vmatrix}$$

$$+ \begin{vmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{vmatrix} + \begin{vmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{vmatrix} + \begin{vmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{vmatrix}$$

permutation of $(1,2,3)$

1 2 3	2 1 3	3 1 2
1 3 2	2 3 1	3 2 1

permutation of $(1, 2, \dots, n)$: there are $n!$ permutations.

each permutation has a sign $+1$ (even) -1 (odd)

- even number of swaps : $+1$ (even)
- odd number of swaps : -1 (odd)

Example

1 2 3
1 3 2 ← odd
3 1 2 ← even.

notation. sign(P) det(P)

permutations ↔ matrices.

$e_1 \; e_2 \; e_3$
$e_1 \; e_3 \; e_2$ ← $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.
$e_3 \; e_1 \; e_2$ ← $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Big formula

$$\det(A) = \sum_{\text{all } P\text{s}} \left(a_{1P(1)} a_{2P(2)} \cdots a_{nP(n)} \right) \det(P).$$