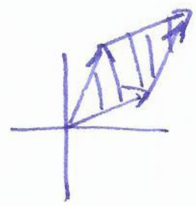
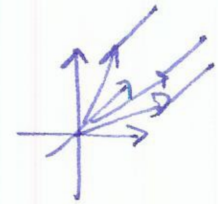
§4.1 Determinants / §4.2

A $n\times n$ matrix $\det(A)$ is a number. $\det(A) \equiv |A|$

- A is invertible iff $\det(A) \neq 0$
- $\det(A)$ = volume of parallelepiped with sides cols of A.





- 2x2 case: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\det(A) = ad - bc$
- $\det(A) = \pm$ product of pivots

key properties:
- $\det(I) = 1$
- row swap changes sign, i.e. if B is A with 2 rows swapped then $\det(A) = -\det(B)$
- $\det(A)$ is linear in each rows.

2x2 case

① $\det I = 1$ $\quad \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 - 0 = 1$

② row exchange $\quad \begin{vmatrix} c & d \\ a & b \end{vmatrix} = cb - ad = -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

③ linear in ~~the~~ each ~~rows~~ rows. individually.

add rows $\quad \begin{vmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}$ check!

scalar mult $\quad \begin{vmatrix} ta & tb \\ c & d \end{vmatrix} = t \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Warning $\quad \det(\underset{n\times n}{tA}) = t^n \det(A)$!

④ if two rows are equal, then $\det(A) = 0$ $\quad \begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$

⑤ subtracting a multiple of one row from another row does not change det.

$$\begin{vmatrix} a - tc & b - td \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix} - t \begin{vmatrix} c & d \\ c & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

⑥ if A has a row of zeros then $\det A = 0$

$$\begin{vmatrix} 0 & 0 \\ c & d \end{vmatrix} = 0$$

⑦ If A is triangular, then $\det A = a_{11} a_{22} \cdots a_{nn}$ product of ~~trian~~ diagonal entries.

$$\begin{vmatrix} a & b \\ 0 & c \end{vmatrix} = ac - 0b = ac.$$

Proof (in general)

$$\det\left(\begin{bmatrix} a_{11} & & \\ 0 & a_{22} & \\ \vdots & \ddots & \\ 0 & \cdots \ 0 & a_{nn} \end{bmatrix}\right) = a_{nn} \det \begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \vdots \\ & & \ddots & 0 \\ 0 & & & 1 \end{bmatrix}$$

$$= a_{nn} a_{n-1\,n-1} \det \begin{bmatrix} a_{11} & & 0 & 0 \\ 0 & a_{22} & \vdots & \vdots \\ & 0 & 1 & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \text{ etc. } \square.$$

⑧ If A is singular then $\det(A) = 0$
If A is invertible then $\det(A) \neq 0$

Proof row reduction gives upper triangular matrix with same det.

$$\begin{bmatrix} \boxed{a_{11}} & & \\ & \boxed{a_{22}} & \\ & & \ddots \ \boxed{a_{nn}} \end{bmatrix}$$ full set of pivots $\Leftrightarrow$ invertible. $\det(A) = a_{11} a_{22} \cdots a_{nn}$.

fewer pivots $\Rightarrow$ zero row $\Rightarrow \det(A) = 0$. $\square$

⑨ $\det(AB) = \det(A)\det(B)$

note if $AA^{-1} = I$ then $\det(A)\det(A^{-1}) = 1 \Rightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$.

⑩ $\det(A^T) = \det(A)$

§4.3 Formulas for the determinant

3x3 case
$$\det \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$$

4x4 case
$$\begin{vmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{vmatrix} = a \begin{vmatrix} f & g & h \\ j & k & l \\ n & o & p \end{vmatrix} - b \begin{vmatrix} e & g & h \\ i & k & l \\ m & o & p \end{vmatrix} + c \begin{vmatrix} e & f & h \\ i & j & l \\ m & n & p \end{vmatrix} - d \begin{vmatrix} e & f & g \\ i & j & k \\ m & n & o \end{vmatrix}$$