§3.1 Orthogonal vectors and subspaces

(44)

$\mathbb{R}^n$: length of a vector $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

$$\|x\|^2 = x_1^2 + x_2^2 + \cdots + x_n^2 = x^T x$$

recall: dot product $x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = x^T y$

$$= \|x\| \, \|y\| \cos\theta$$

two vectors are orthogonal iff $x \cdot y = 0$

A set of vectors $\{v_1, \ldots, v_k\}$ is orthogonal if $v_i \cdot v_j = 0$ for all $i \neq j$.

Lemma orthogonal vectors are linearly independent

Proof suppose $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$

then $(c_1 v_1 + \cdots + c_k v_k) \cdot v_i = 0 \cdot v_i = 0$

$$c_1 \underbrace{v_1 \cdot v_i}_{0} + \cdots + c_i \underbrace{v_i \cdot v_i}_{\|v_i\|^2 > 0} + \cdots + c_k \underbrace{v_k \cdot v_i}_{0}$$

$$c_i \|v_i\|^2 = 0 \implies c_i = 0 \implies \text{all } c_i = 0 \ \square.$$

Special bases $\{v_1, \ldots, v_k\}$ orthogonal basis if $v_i \cdot v_j = 0 \quad i \neq j$.

$\{v_1, \ldots, v_k\}$ orthonormal basis if $v_i \cdot v_j = 0 \quad i \neq j$ and $\|v_i\| = 1$

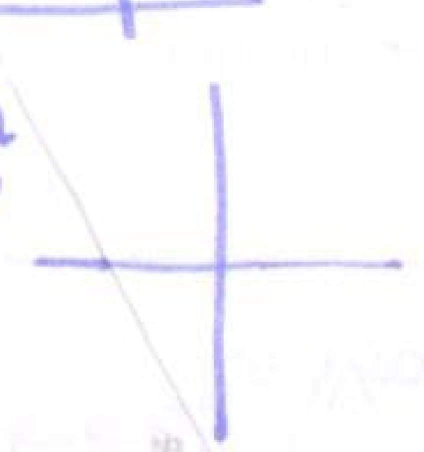
Examples $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ $\quad \left\{ \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \right\}$

Orthogonal subspaces

V, W subspaces, are orthogonal if $v \cdot w = 0$ for all $v \in V, w \in W$.

Examples $\{0\}$ orthogonal to every subspace. non-example

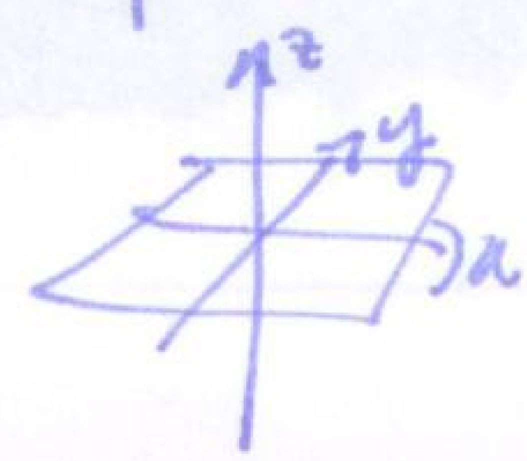
$\mathbb{R}^2$

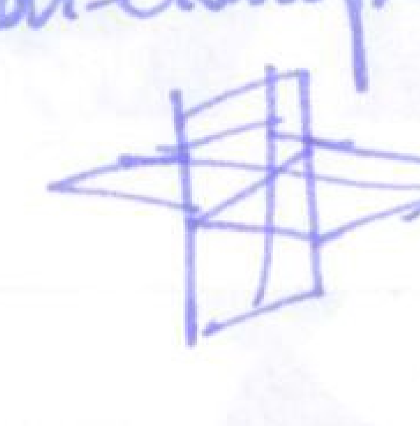


V basis $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

W basis $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

$\mathbb{R}^3$





Thm 3C (Fundamental theorem of orthogonality) A $m \times n$

The row space is orthogonal to the null space/kernel ($\in \mathbb{R}^n$)
The column space is orthogonal to the left null space/cokernel ($\in \mathbb{R}^m$)

Proof ① suppose $x \in N(A)$ then $Ax = 0$ $A\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

i.e. $\underline{x}$ is orthogonal to all the rows of A, and so all linear combinations of the rows $(a_1 r_1 + a_2 r_2 + \cdots + a_m r_m) \cdot x = a_1 \underbrace{r_1 \cdot x}_{0} + a_2 \underbrace{r_2 \cdot x}_{0} + \cdots + a_m \underbrace{r_m \cdot x}_{0} = 0$

so $N(A) \perp Row(A)$.

spose $x \in N(A^T)$ so $x^T A = 0$ $[x_1 \cdots x_n] A = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$

i.e. x is orthogonal to the columns of A, and so all linear combinations of the cols., so $N(A^T) \perp col(A)$.

② suppose $x \in N(A)$ so $Ax = 0$
$v \in Row(A)$ so $v = A^T z$ for some z.

so $v^T x = (A^T z)^T x = z^T A x = z \cdot 0 = 0$. □

Defn let $V \subset \mathbb{R}^n$ be a subspace, then the collection of all vectors perpendicular to V is a subspace called the orthogonal complement $V^\perp$ say "V perp"

Thm 3D (Fundamental theorem of linear algebra, part II)

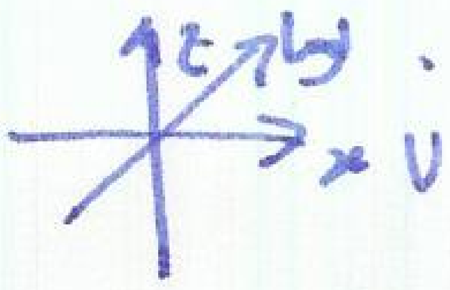
$N(A) = row(A)^\perp$ $(\subset \mathbb{R}^n)$
$N(A^T) = col(A)^\perp$ $(\subset \mathbb{R}^m)$

Proof recall: $\left.\begin{array}{l} \dim(N(A)) = n - r \\ \dim(row(A)) = r \end{array}\right\}$ orthogonal vectors independent!
so $\dim(span(N(A), row(A))) \geq n$

similarly: $\left.\begin{array}{l} \dim(N(A^T)) = m - r \\ \dim(col(A)) = r \end{array}\right\}$ so span all of $\mathbb{R}^m$.

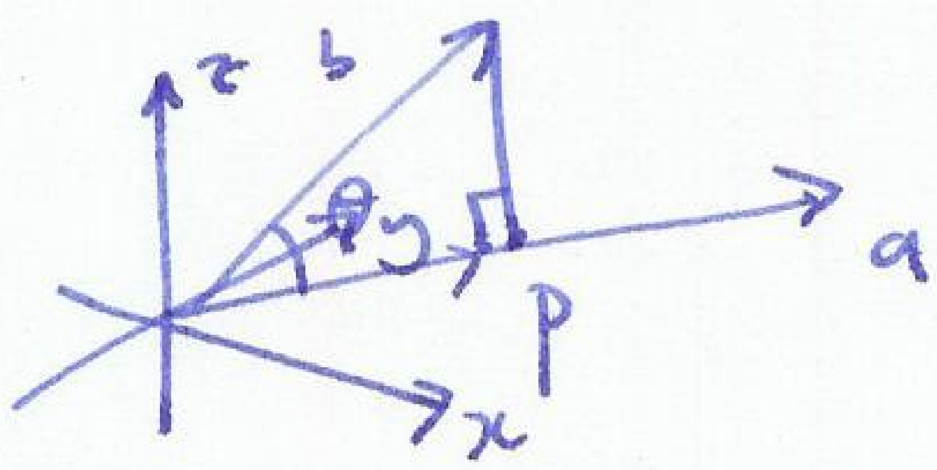
in each case any orthogonal vector to $row(A)$ / $col(A)$ must be in $N(A)$ / $N(A^T)$. □

Remark: V, W can be perpendicular, but not span $\mathbb{R}^n$.



but if $V = W^\perp$ then $\dim(V) + \dim(W^\perp) = n$.

§3.2 Projections



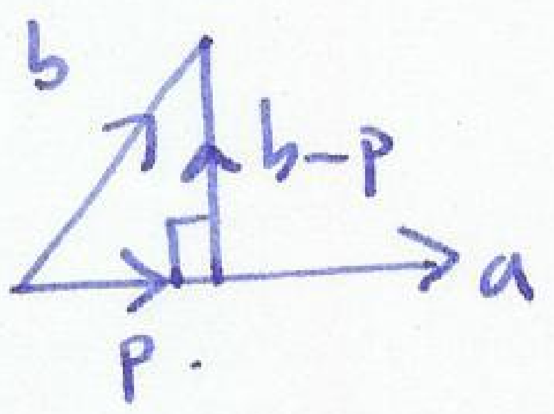
p = projection of b onto line through a.

Recall: geometric def^n of dot product

$$a^Tb = \|a\|\,\|b\|\cos\theta$$

Key fact: $$p = \frac{a^Tb}{a^Ta}\,a = \frac{a.b}{a.a}\,a$$

Proof: $p = \lambda a$ for some λ



$$b - p \cdot a = 0$$

$$(b - (\lambda a)).a = 0$$

$$a.b - \lambda a.a = 0$$

$$\lambda = \frac{a.b}{a.a} \quad \text{so} \quad p = \frac{a.b}{a.a}\,a \quad \square.$$

Corollary: Schwarz inequality

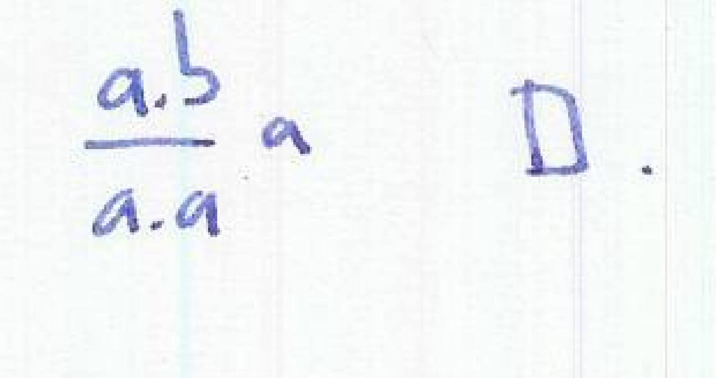
$$|a^Tb| = |a.b| \le \|a\|\,\|b\|$$

with equality iff $a = \lambda b$.

Example: project $b = (1,2,3)$ onto $a = (3,2,1)$

$$p = \frac{a.b}{a.a}\,a = \frac{3+4+3}{9+2+1}(3,2,1) = \frac{10}{12}(3,2,1).$$

Observation: projection is a linear map!



$\mathbb{R}^n \longrightarrow$ line through a $\subset \mathbb{R}^n$

$$\frac{(x+y).a}{a.a}\,a = \frac{x.a}{a.a}\,a + \frac{y.a}{a.a}\,a.$$

$$\frac{\lambda x.a}{a.a}\,a = \lambda\frac{x.a}{a.a}\,a.$$

Q: what is the matrix P corresponding to projection?

$$P: x \longmapsto a\,\frac{a^T x}{a.a} \quad \text{so } P = \frac{aa^T}{a.a} \qquad \underbrace{\underset{n\times 1}{a}\;\underset{1\times n}{a^T}}_{n\times n}.$$

$$\underbrace{\underset{1\times n}{a^T}\;\underset{n\times 1}{a}}_{1\times 1}.$$

Example project onto $(1,1,1)$

$$P = \frac{1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}\begin{bmatrix}1 & 1 & 1\end{bmatrix} = \frac{1}{3}\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$$

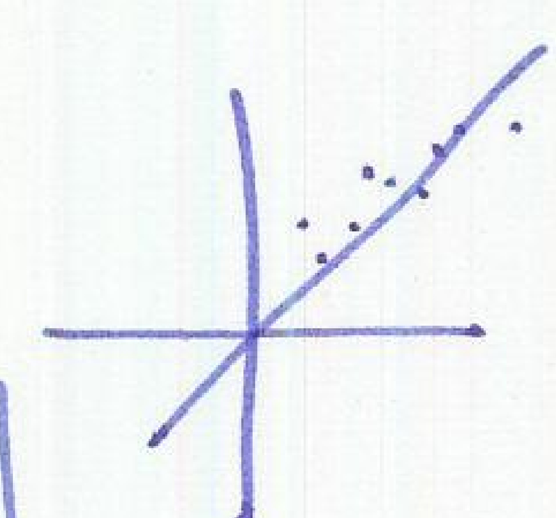
Properties of projections
1. P is symmetric $\quad \left(\frac{aa^T}{a.a}\right)^T = \frac{(a^T)^T a^T}{a.a} = \frac{aa^T}{a.a}.$
2. $P^2 = P$

Remark (3J) A^T can be defined as: the matrix s.t.

$$Ax . y = x . A^T y$$

i.e. $(Ax)^T y = x^T A^T y = x^T (A^T y).$

§3.3 Projections and least squares



Inconsistent equations: $\quad 2x = b_1 \quad a^T x = b \quad b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$

$3x = b_2 \quad a = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

$4x = b_3$

find x which minimizes error squared: $E^2 = (2x-b_1)^2 + (3x-b_2)^2 + (4x-b_3)^2$

$$\frac{dE^2}{dx} = 2\left[(2x-b_1)2 + (3x-b_2)3 + (4x-b_3)4\right] = 0$$

minimum at $\hat{x} = \dfrac{2b_1 + 3b_2 + 4b_3}{2^2+3^2+4^2} = \dfrac{a^T b}{a^T a}$ ← least squares solution.

$$\boxed{\hat{x} = \frac{a^T b}{a^T a}}$$

General case $\quad ax = b$

~~Note: error~~ minimize $E^2 = \|ax-b\|^2 = (a_1 x - b_1)^2 + \cdots + (a_m x - b_m)^2$

$$\frac{dE^2}{dx} = 2\left[(a_1 x - b_1)a_1 + \cdots + (a_m x - b_m)a_m\right] = 0 \qquad \hat{x} = \frac{a^T b}{a^T a}$$

note error $ax - b \perp$ to a! $\quad a.(ax-b) = a.a\,\frac{a.b}{a.a} - a.b = 0.$