$T: V \to W$

choose a basis $v_1, \ldots, v_m$ ~~$x_1, \ldots, x_m$~~ for V then $T(v_i) \in W$ so

~~$y_1, \ldots, y_n$~~ for W

$w_1, \ldots, w_n$

$$T(v_1) = a_{11} w_1 + a_{12} w_2 + \cdots + a_{1n} w_n$$

so wrt to basis $\{v_1, \ldots, v_m\}$ for V

$$T(v_2) = a_{21} w_1 + a_{22} w_2 + \cdots + a_{2n} w_n$$

$\{w_1, \ldots, w_n\}$ for W

$$T(v_m) = a_{m1} w_1 + a_{m2} w_2 + \cdots + a_{mn} w_n$$

T is represented by $A = [a_{ij}]$

<u>claim</u>: knowing values of T on a basis determines T.

<u>proof</u> Let $x \in V$, then $x = c_1 v_1 + c_2 v_2 + \cdots + c_m v_m$ (uniquely!)

so
$$\begin{aligned} T(x) &= T(c_1 v_1 + c_2 v_2 + \cdots + c_m v_m) \\ &= T(c_1 v_1) + T(c_2 v_2) + \cdots + T(c_m v_m) \\ &= c_1 T(v_1) + c_2 T(v_2) + \cdots + c_m T(v_m) \quad \square. \end{aligned}$$

if we write x in basis $\{v_1, \ldots, v_m\}$ then $x = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}$ and $T(x) = Ax \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$.

Note $Tv_1 = T\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = $ 1st col of A

$Tv_i = T\begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix} = i$th col of A

<u>Examples</u> P_n = degree n polynomials

· <u>differentiation</u> $P_n \to P_{n-1}$

$p(t) \mapsto p'(t)$ linear: $\frac{d}{dt}(cp(t)) = c\frac{dp}{dt}$

$$\frac{d}{dt}(p+q) = \frac{dp}{dt} + \frac{dq}{dt}$$

$P_3 \to P_2$

$$a_3 t^3 + a_2 t^2 + a_1 t + a_0 \longmapsto 3a_3 t^2 + 2a_2 t + a_1$$

bases: $\{t^3, t^2, t, 1\}$ $\{t^2, t, 1\}$

$t^3 \longmapsto 3t^2$

$t^2 \longmapsto 2t$

$t \longmapsto 1$

$1 \longmapsto 0$

$$A = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

integration (with additive constant equal to zero)

$P^n \longrightarrow P^{n+1}$

$p(t) \mapsto \int p(t)\,dt$

$$a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0 \mapsto \frac{a_n t^{n+1}}{n+1} + \frac{a_{n-1} t^n}{n} + \cdots + \frac{a_1 t^2}{2} + a_0 t$$

$P^2 \longrightarrow P^3$

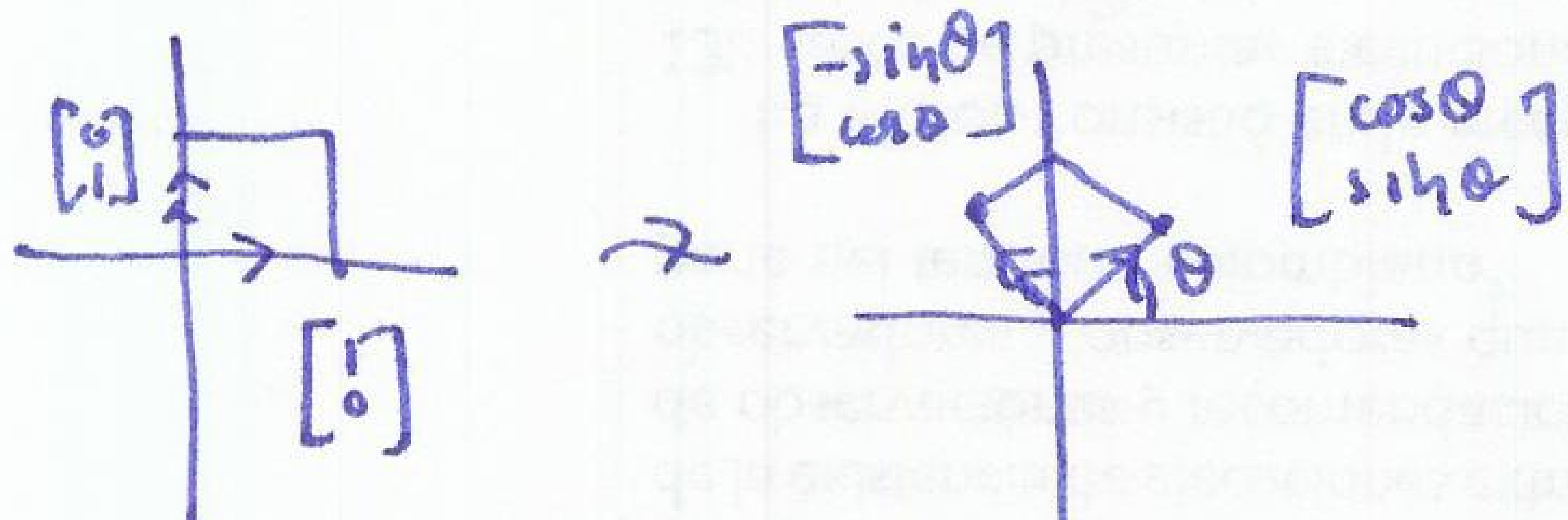
$$a_2 t^2 + a_1 t + a_0 \mapsto \frac{a_3 t^3}{3} + \frac{a_2 t^2}{2} + a_0 t \qquad A = \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$t^2 \longmapsto \frac{1}{3} t^3$

$t \longmapsto \frac{1}{2} t^2$

$1 \longmapsto t$

Rotations



$R_\theta : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Note rotate twice: $R_\theta R_\theta = R_\theta^2$ should equal $R_{2\theta}$.

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta - \sin^2\theta & -2\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & -\sin^2\theta + \cos^2\theta \end{bmatrix} = \begin{bmatrix} \cos 2\theta & -\sin 2\theta \\ \sin 2\theta & \cos 2\theta \end{bmatrix} \checkmark$$

Exercise: $R_\theta^{-1} = R_{-\theta} \qquad R_\theta R_\phi = R_{\theta+\phi}$.

General fact

$U \xrightarrow{S} V \xrightarrow{T} W$ linear maps

pick bases $\{u_1 \ldots u_{\cancel{m}\, l}\}$ $\{v_1 \ldots v_m\}$ $\{w_1 \ldots w_n\}$.

then $\quad A \quad B$

$\cancel{l \times m}$ $m \times l$ $\quad n \times m$

$x \longmapsto S(x) \longmapsto T(S(x))$

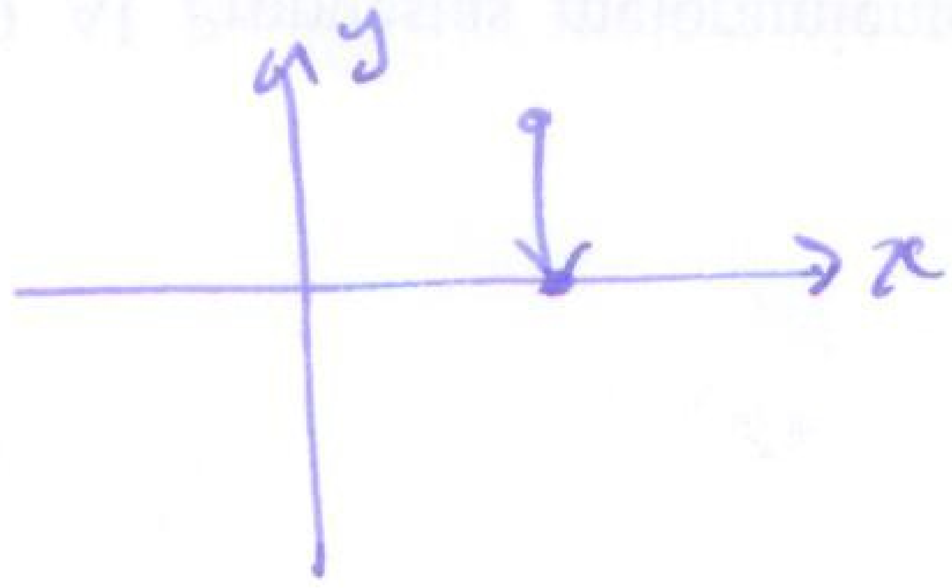
$x \longmapsto Ax \qquad BAx$

$l\times 1 \quad \underbrace{m\times l \;\; l\times 1}_{m\times 1} \quad \underbrace{n\times m \;\; m\times l \;\; l\times 1}_{n\times 1}$.

so composition of linear maps corresponds to matrix multiplication

Projections

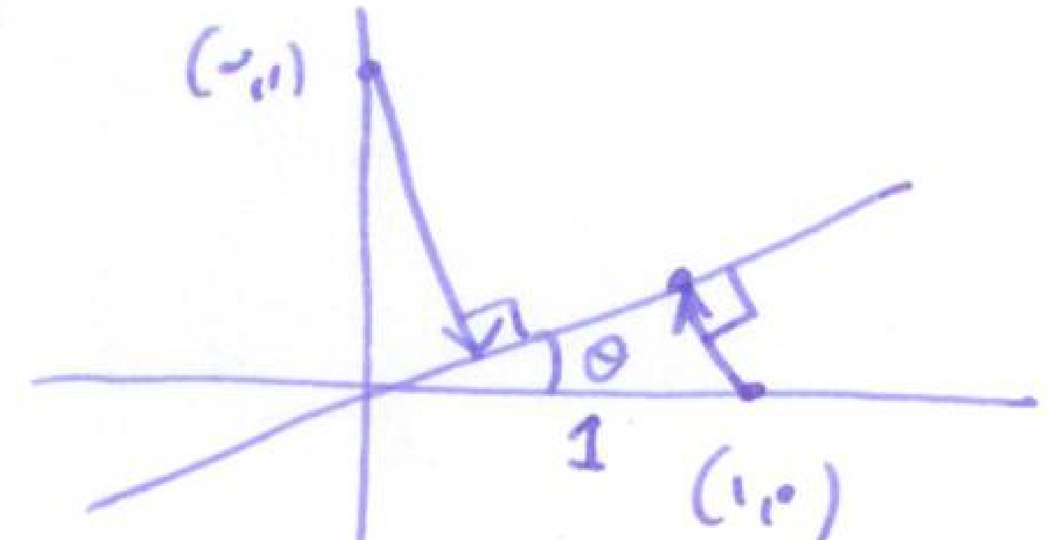
Example $\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\0\end{bmatrix}$

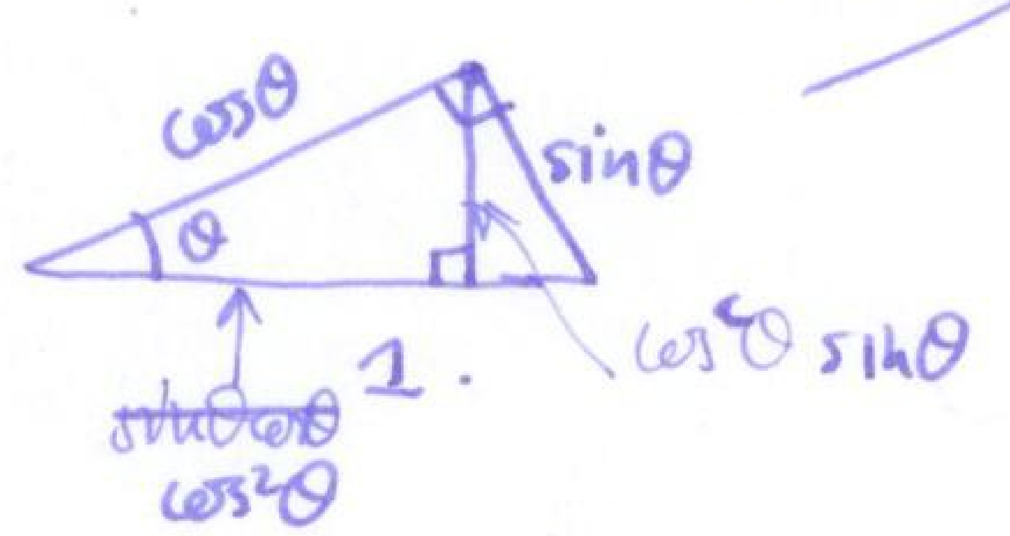


not invertible!

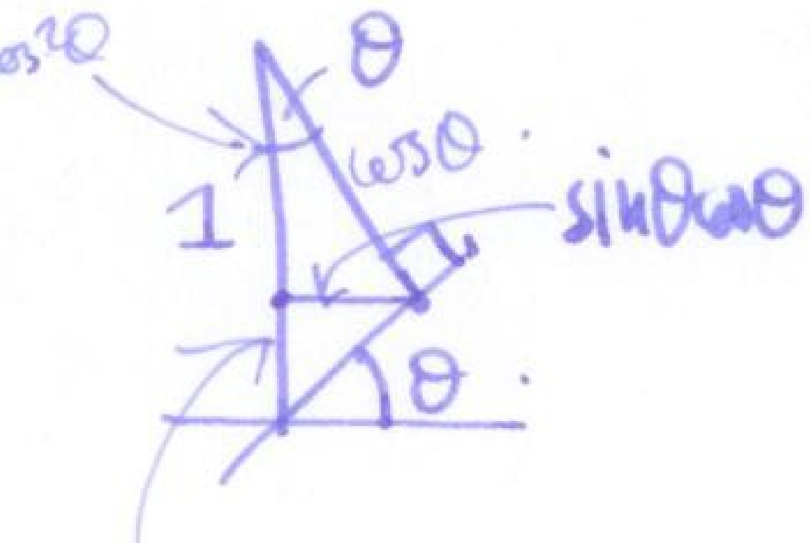
projection onto line at angle θ

$\begin{bmatrix}1\\0\end{bmatrix}\mapsto\begin{bmatrix}\cos^2\theta\\ \sin\theta\cos\theta\end{bmatrix}$.





$\begin{bmatrix}0\\1\end{bmatrix}\mapsto\begin{bmatrix}\cos\theta\sin\theta\\ \sin^2\theta\end{bmatrix}$



$1-\cos^2\theta=\sin^2\theta$

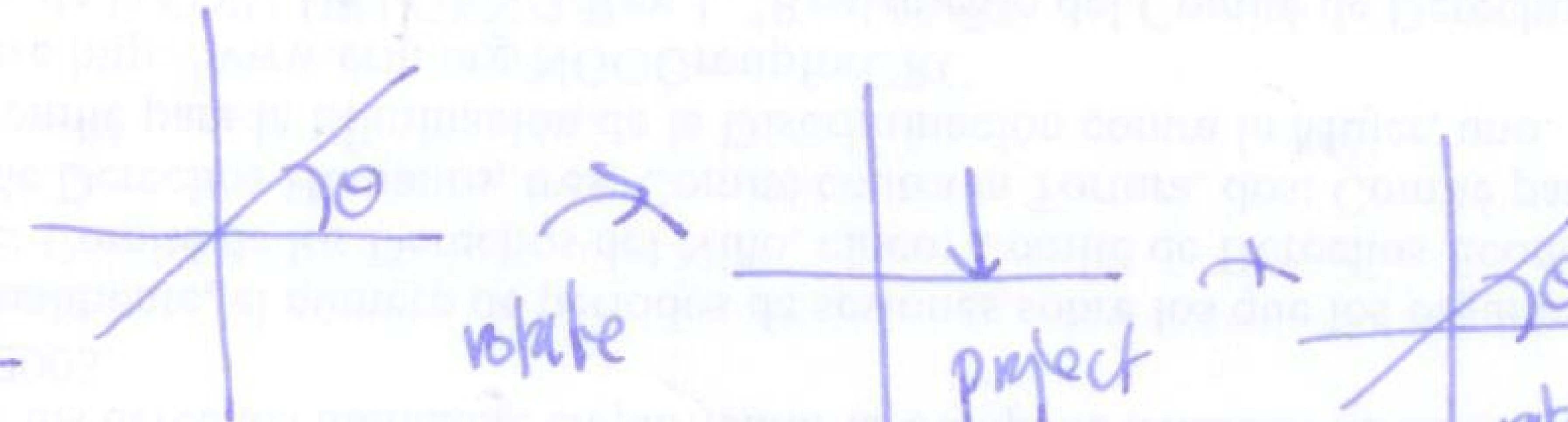
$$P=\begin{bmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}.$$

Observation $P^2=P$

check $\begin{bmatrix}c^2 & sc\\ sc & s^2\end{bmatrix}^2=\begin{bmatrix}c^4+c^2s^2 & sc^3+s^3c\\ sc^3+s^3c & s^2c^2+s^4\end{bmatrix}$

$$=\begin{bmatrix}c^2(c^2+s^2) & sc(c^2+s^2)\\ sc(c^2+s^2) & s^2(c^2+s^2)\end{bmatrix}.$$

Better Reflections



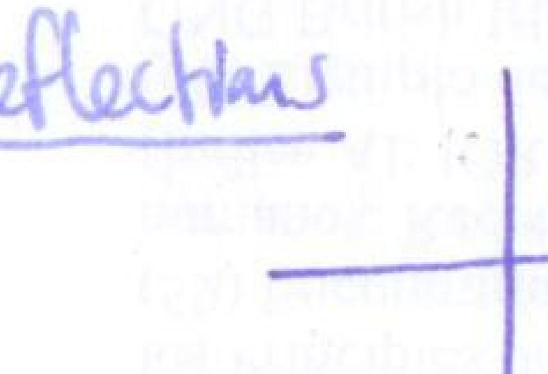
$R_{-\theta}=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$ $\quad\underset{P}{\begin{bmatrix}1&0\\0&0\end{bmatrix}}\quad$ $\underset{R_\theta}{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}.$

$$R_\theta P R_{-\theta}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta\\ 0 & 0\end{bmatrix}.$$

$$=\begin{bmatrix}\cos^2\theta & \cos\theta\sin\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{bmatrix}$$

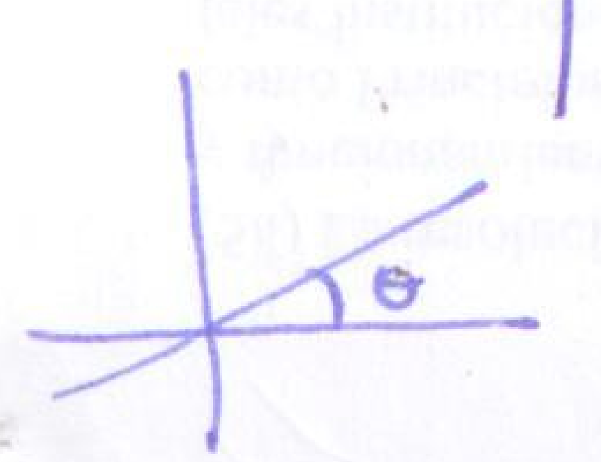
Reflections

$\begin{bmatrix}x\\y\end{bmatrix}\to\begin{bmatrix}-x\\y\end{bmatrix}$



$H=\begin{bmatrix}-1&0\\0&1\end{bmatrix}$ note $H^2=I$.

check $H_\theta^2=I$!



$$H_\theta=R_\theta H R_{-\theta}=\begin{bmatrix}2c^2-1 & 2cs\\ 2cs & 2s^2-1\end{bmatrix}$$