Column space $C(A)$

$$\underbrace{\underset{m\times n}{A}\ \underset{n\times 1}{x}}_{m\times 1} \qquad A: \mathbb{R}^n \to \mathbb{R}^m$$

<u>note</u>: $C(A)$ is also the <u>image</u> of A.

$= \{ Ax \in \mathbb{R}^m \mid x \in \mathbb{R}^n \}$. as $\begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = c_1x_1 + c_2x_2 + \cdots + c_nx_n$.

<u>Warning</u> Row operations <u>change</u> the column space! <u>Example</u> $\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}} \rightsquigarrow \underset{U}{\begin{bmatrix} \boxed{1} & 1 \\ 0 & 0 \end{bmatrix}}$.

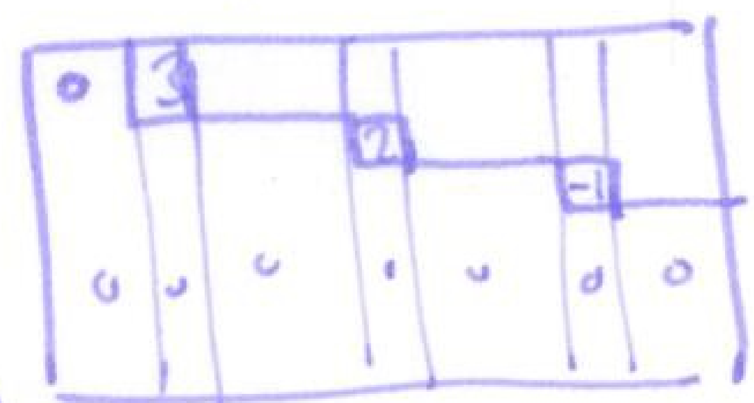
pivot column ↑ ↑ free var column

<u>Claim</u>: the columns of A corresponding to pivots in U are a basis for $C(A)$. so $\dim(C(A)) = \#\text{pivots} = r = \text{rank}(A)$ $(= \dim \text{Row}(A))$.

<u>Proof</u> observation: $Ax = 0 \Leftrightarrow Ux = 0$ $(N(A) = N(U))$.

a linear dependence between the columns of A is $[c_1 \cdots c_n]$ $a_1c_1 + a_2c_2 + \cdots + a_nc_n = 0$

i.e. $\begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}\begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix} = 0$ i.e. $Ax = 0$, but then $Ux = 0$, so this gives linear dependence between columns of U and vice versa.



The pivot columns of U are a basis for $C(U)$:

$\Rightarrow$ the corresponding columns of A are a basis for $C(A)$. □.

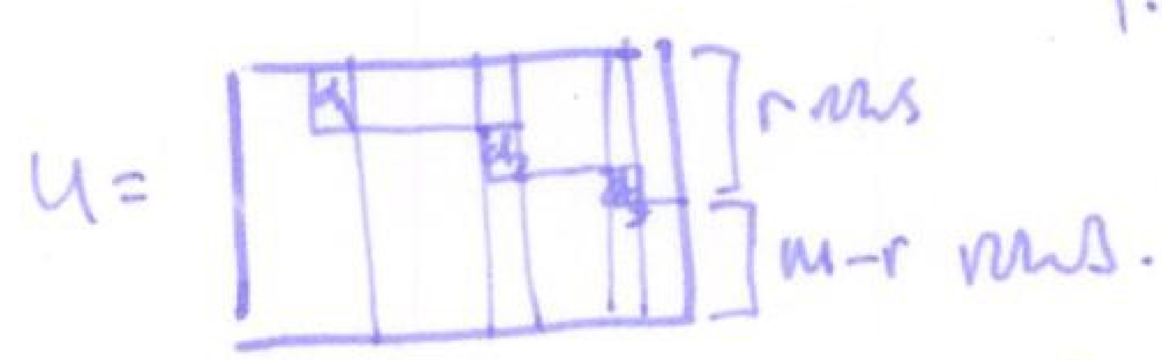
Left nullspace $N(A^T)$

i.e. $y^TA = 0$ $\qquad \begin{bmatrix} y_1 & y_2 & \cdots & y_m \end{bmatrix}\begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_m \end{bmatrix} = y_1r_1 + y_2r_2 + \cdots + y_mr_m = 0$

i.e. linear combinations of rows which give 0.

recall: $PA = LU$

$\underbrace{L^{-1}P}\,A = U$



bottom $m-r$ rows of $L^{-1}P$ are a basis for $N(A^T)$.

alternate method: just solve $A^Ty = 0$.

Fundamental theorem of linear algebra part I

$\dim(C(A)) = r$

$\dim(N(A)) = n - r$

$\dim(\text{Row}(A)) = r$

$\dim(N(A^T)) = m - r$

Existence of inverses

recall if A has a left inverse and a right inverse, they are equal.

fact: A has left and right inverses iff A is square $n\times n$ and $\text{rank}(A)=n$.

One sided inverses.

$\Leftrightarrow$ can always solve $Ax=b$

Example

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \\ * & * \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2×3 $\quad$ 3×2 $\quad$ not unique!

Fact A $m\times n$ $\quad$ $\text{rank}(A)=r$

if $r=m$ then there is a right inverse $A^T(AA^T)^{-1}$ check!

$r=n$ then there is a left inverse $(A^TA)^{-1}A^T$ check!

problem: when is AA^T invertible? $\quad$ $\underset{m\times n}{A}\underset{n\times m}{A^T}$ $\quad$ $m\times m$ $\quad$ rank $\leq r$

A^TA $\quad$ $\underset{n\times m}{A^T}\underset{m\times n}{A}$ $\quad$ $n\times n$ $\quad$ rank $\leq r$.

Two sided inverses

$\underset{m\times n}{A}$ has a two sided inverse iff $m=n=r=\text{rank}(A)$

equivalent conditions:
- columns of A span $\mathbb{R}^n$, so $Ax=b$ has at least one solution for every b.
- the columns of A are independent, so $Ax=0$ has unique soln $x=0$
- the rows of A span $\mathbb{R}^n$
- the rows are linearly independent
- elimination gives $PA=LDU$ with n pivots.
- $\det(A)\neq 0$

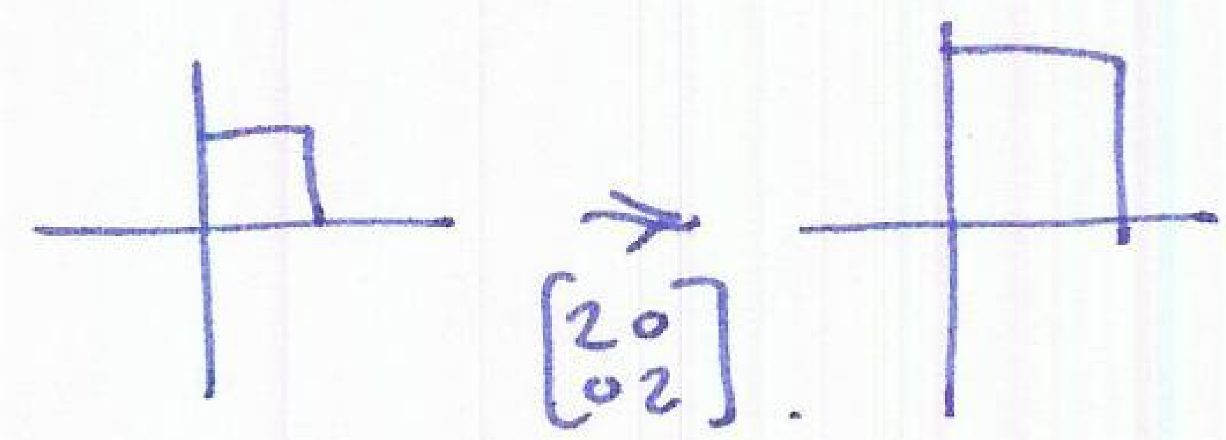
§2.6 Linear transformations

A $m \times n$ matrix $\quad A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$

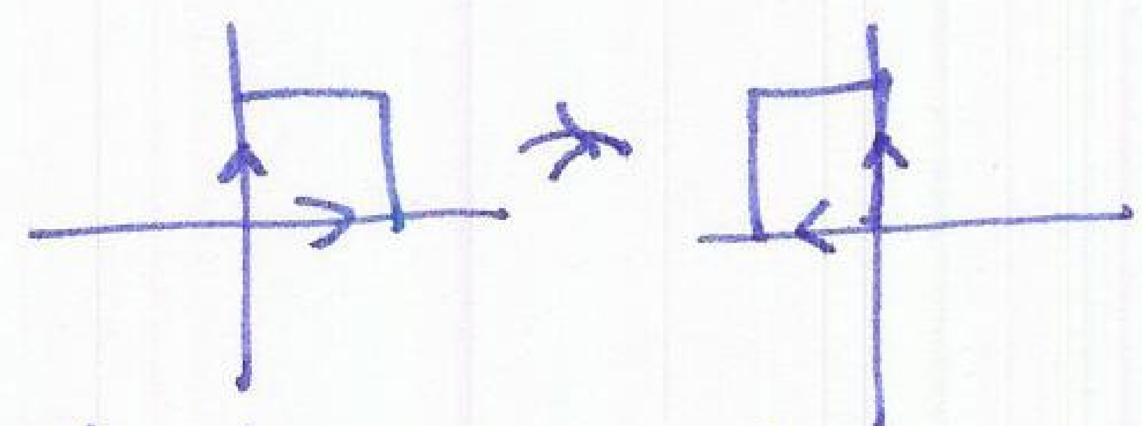
$$\underbrace{x}_{n\times 1} \longmapsto \underbrace{Ax}_{m \times 1}$$

Examples

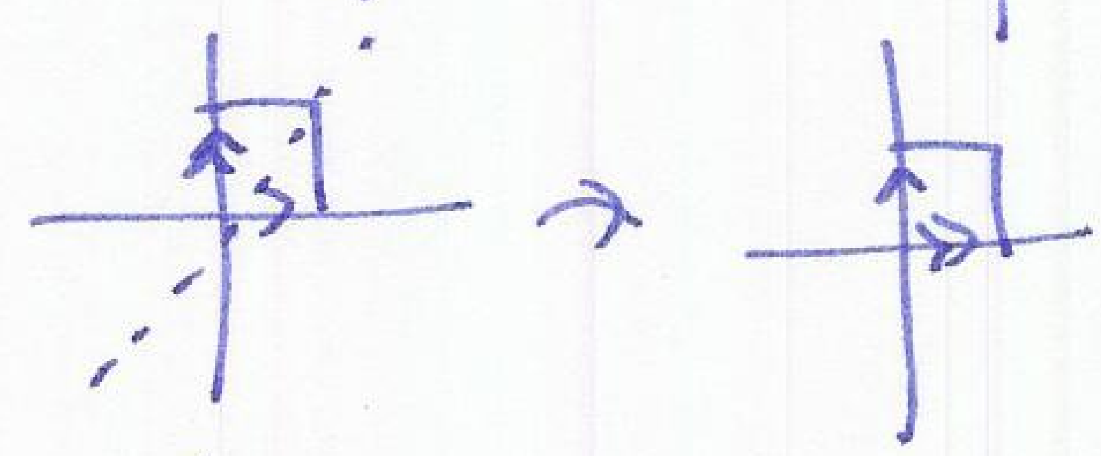
- $A = I \quad \underline{x} \mapsto \underline{x}$ identity.
- $A = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix} = cI$ expansion/contraction $c > 1 \quad 0 < c < 1$



- $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ rotation $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} -y \\ x \end{bmatrix}$

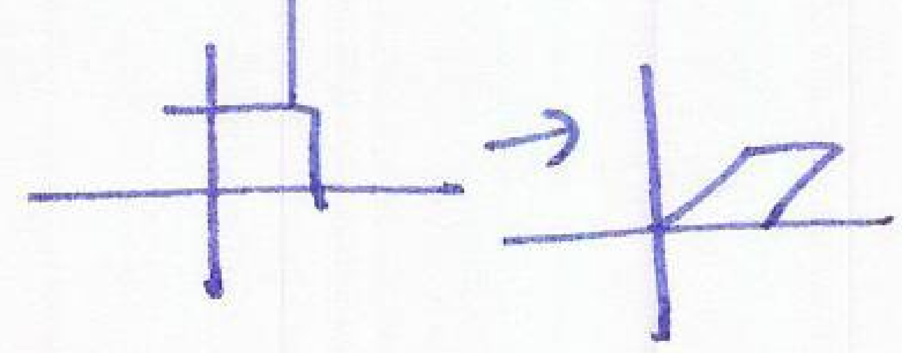


- $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ reflection $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} y \\ x \end{bmatrix}$



- $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ projection $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \\ 0 \end{bmatrix}$

- $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ shear $\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x+y \\ y \end{bmatrix}$



Observations $A : \mathbb{R}^n \to \mathbb{R}^m$

- $A0 = 0$ always sends zero vector to zero vector
- $A(cx) = c(Ax)$
- $A(x+y) = Ax + Ay$

V, W vectorspaces.

Def^{n} A map $T : \underset{V}{\mathbb{R}^n} \to \underset{W}{\mathbb{R}^m}$ is a <u>linear map/transformation</u> if

- $T(cx) = c(Tx)$
- $T(x+y) = Tx + Ty$

Examples

- $A : \mathbb{R}^m \to \mathbb{R}^m$
  $x \mapsto Ax$

Fact: every linear map can be described as a matrix.