§2.4 The four fundamental subspaces

A $m\times n$ matrix of $\text{rank}(A)=r$ (# pivots in row-echelon form).

- The column space $C(A)$. $\dim(C(A)) = r$ $\quad C(A) \subset \mathbb{R}^m$
  all linear combinations of the columns of A.
- The nullspace (kernel) $N(A)$. All vectors x s.t. $Ax=0$ $\quad N(A) \subset \mathbb{R}^n$
  $\dim(C(A)) = n-r$
- The row space $Row(A) = C(A^T)$ spanned by rows of A $\quad Row(A) \subset \mathbb{R}^n$
  $\dim(Row(A)) = r$
- The left nullspace / cokernel $N(A^T)$. All vectors y s.t. $A^Ty=0$ $\quad y^TA=0$ Rows $\quad N(A^T) \subset \mathbb{R}^m$
  $\dim(N(A^T)) = m-r$

Example

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} \boxed{1} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \qquad C(A) \text{ spanned by } \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad \dim r = 1.$$

$\text{rank}(A) = r$. $\quad 2\times 3$

$$N(A) \quad s\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{spanned by } \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$\dim \underbrace{3-1}_{n-r} = 2$.

$Row(A)$ spanned by $[1\ 0\ 0]$ $\quad \dim r = 1$.

cokernel $N(A^T)$ spanned by $[0\ 1]$ $\quad \dim m-r = 2-1 = 1$.

General case (how to find bases for the spaces)

Row space

Observation: row operations do not change the row space, because they are reversible.

$$\begin{matrix} ① \\ ② \\ ③ \end{matrix}\begin{bmatrix} A \end{bmatrix} \rightsquigarrow \begin{matrix} ① \\ ②+① \\ ③-4① \end{matrix}\begin{bmatrix} B \end{bmatrix}$$

row space: linear combinations of rows 1, 2, 3.

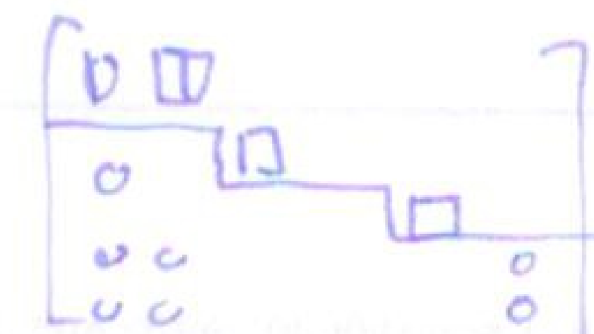
row space, linear combinations of ①, ②+①, ③-4①.
so contained in $Row(A)$, $\quad Row(B) \subset Row(A)$.
but, ①, ②, ③ linear combination of rows of B,
so $Row(A) \subset Row(B)$.

$\Rightarrow Row(A) = Row(B)$

this shows $A \rightsquigarrow U$ ref(A) then $\boxed{Row(A) = Row(U)}$.

observation: the pivot rows of U form a basis for $Row(U) = Row(A)$.

Example $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$\qquad A \qquad\qquad\qquad\qquad U$

general $A \rightsquigarrow$ 

check: they span, zero rows contribute nothing

- they are linearly independent. suppose $c_1 r_1 + c_2 r_2 + \cdots + c_k r_k = 0$
r_1 contains pivot with zeros below, $\Rightarrow c_1 = 0$ then r_2 contains pivot with zeros below and above so $c_2 = 0$ etc.

$\dim(Row(A)) = \dim(Row(U)) = \text{rank}(A) = r$

Null space / kernel

observation row operations do not change the solution set to the equations, because they are reversible.

$Ax = 0 \quad \begin{bmatrix} r_1 \\ r_2 \\ r_3 \end{bmatrix} x = \begin{bmatrix} r_1 x \\ r_2 x \\ r_3 x \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad B = \begin{bmatrix} r_1 \\ r_2 - r_1 \\ r_3 + 4r_1 \end{bmatrix} x = \begin{bmatrix} r_1 x \\ (r_2 - r_1)x \\ (r_3 + 4r_1)x \end{bmatrix} = 0$

or $L_1 A = B \quad L_1$ invertible $A = L_1^{-1} B$.

so $Bx = L_1 A x = 0 \qquad Bx = 0 \Rightarrow L_1^{-1} B x = 0 \Rightarrow Ax = 0$

$N(A) \subset N(B) \qquad$ so $N(B) \subset N(A)$

$\Rightarrow N(A) = N(B)$.

observation: to find a basis for $N(A)$ do back substitution in U.

get a basis vector for each free variable, so $\dim(N(A)) = \#\text{cols} - \text{rank}(A)$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad n - r$

recall in example $s \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$. they span as they give all solutions.

they are linearly indep as they are upper triangular.