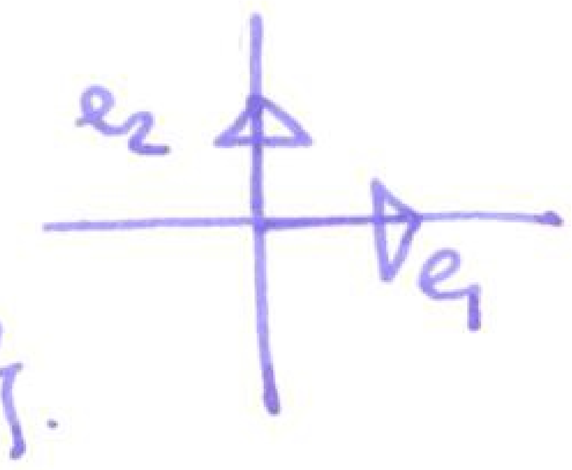
<u>Examples of bases</u> · $\mathbb{R}^2$ $\{e_1, e_2\} = \{[1,0], [0,1]\}$



· any pair of vectors which span $\mathbb{R}^2$, i.e. are not parallel.

eg $\{[1,0],[1,1]\}$ $\{[2,3],[4,-1]\}$.

there are infinitely many different bases for $\mathbb{R}^2$. (a basis is <u>not</u> unique).

· $\mathbb{R}^n$ $\{e_1, \ldots, e_n\}$.

<u>Non-examples</u>: any set containing 0 zero vector.

<u>Def^n</u> Any two bases for V have the same number of vectors. The number of vectors in a basis is called the <u>dimension</u> of V.

<u>Example</u> $\dim(\mathbb{R}^2) = 2$

$\dim(\mathbb{R}^n) = n$

<u>Th^m</u> (23.2k) Let V be a vector space with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$

Then $m = n$.

<u>Proof</u> spose $\cancel{k > k}$ $n > m$. The v_i are a basis, so the span V, so every w_j is a linear combination of the v_i, i.e.

$$w_1 = a_{11} v_1 + a_{21} v_2 + \cdots + a_{m1} v_m$$
$$w_2 = a_{12} v_1 + a_{22} v_2 + \cdots + a_{m2} v_m$$
$$\vdots$$
$$w_n = a_{1n} v_1 + a_{2n} v_2 + \cdots + a_{mn} v_m.$$

let $W = \begin{bmatrix} w_1 \, w_2 \cdots w_n \end{bmatrix}$ $V = \begin{bmatrix} v_1 \cdots v_n \end{bmatrix}$

then $W = VA$

$$A = \begin{array}{cc} m & \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix} \\ & n \end{array}.$$

A has at most m pivots. as $n > m$

there is a free variable after row-reducing $\Rightarrow$ non trivial solution to $A\underline{x} = \underline{0}$

i.e. $Wx = 0 \Rightarrow x_1 w_1 + x_2 w_2 + \cdots x_n w_n = 0$ not all x_i zero

$\Rightarrow$ w_i linearly dependent $\#$. so $m < n$ $\square$.

Thm (2.3.22). Any linearly independent set can be extended to a basis, by adding more vectors if necessary

· Any spanning set can be reduced to a basis, by discarding vectors, if necessary

Remarks. A basis is a maximal independent set.

It is also a minimal spanning set

· If V has dimension k: at most k vectors may be linearly independent

at least k vectors are needed to span the space.

Examples:

Quadratic polynomials $\quad ax^2+bx+c \quad$ basis $\{1, x, x^2\}$ dim 3.

2x2 matrices $\quad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad$ basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

§2.4 The four fundamental subspaces

Given a matrix A there are 4 natural subspaces built from $\underset{m\times n}{A}$.

· The column space $C(A)$ spanned by columns of A. $\dim(C(A)) = \underset{r}{\mathrm{rank}(A)}$

· The null space (kernel) $N(A) = \{x \mid Ax = 0\}$ dimension $n-r$

· The row space spanned by rows of A, dimension $r = \mathrm{rank}(A)$ $(C(A^T))$.

· The left nullspace (cokernel), all vectors y s.t. $y^T A = 0$, dimension $m-r$ $(N(A^T))$.

$(y^T A)^T = A^T y$.

$\underset{m\times n}{A}$ nullspace $N(A)$, row space $C(A^T)$ subspaces of $\mathbb{R}^n$

left nullspace $N(A^T)$, columnspace $C(A)$ subspaces of $\mathbb{R}^m$