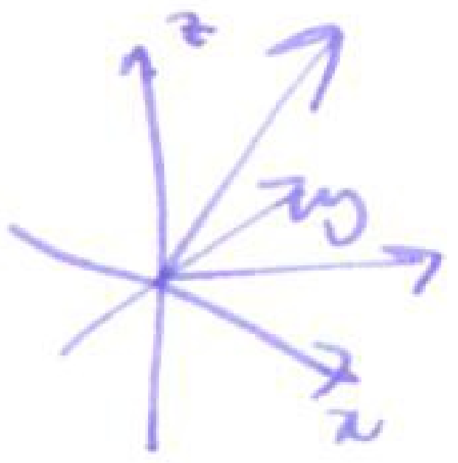
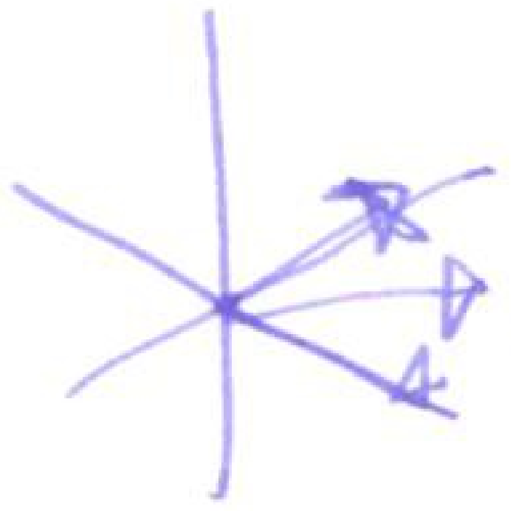
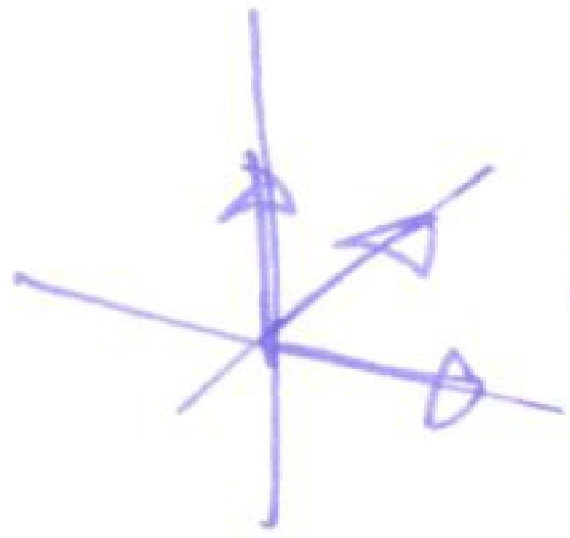


independent.



dependent.



independent.

<u>Example</u> $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix}$ columns are dependent : $C2 = 3C1$

rows are dependent $R3 = 2R2 - 5R1$

<u>Example</u> $\begin{bmatrix} 3 & 4 & 2 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$ columns are independent.

rows are independent.

Q: how do we check this? $C(A)$ column space. $A = \begin{bmatrix} c_1 & c_2 \cdots & c_n \end{bmatrix}$.

are the columns dependent? want : $a_1c_1 + a_2c_2 + \cdots + a_nc_n = 0$

i.e. solve $Ax = 0$: if only solution is $x = 0$: columns are <u>independent</u> (i.e. $N(A) = \{0\}$)

if any non-zero solution : columns are <u>dependent</u>. $N(A)$ bigger than $\{0\}$.

row reduce:

Ⓐ $\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \boxed{1} & 3 & 3 & 2 \\ 0 & 0 & \boxed{3} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ all cols can be made from sums of ~~a~~ pivot cols. ! add pivot cols are independent, extra cols get dependent vectors.

Ⓑ $\begin{bmatrix} \boxed{3} & 4 & 2 \\ 0 & \boxed{1} & 5 \\ 0 & 0 & \boxed{2} \end{bmatrix}$ 3 pivots $N(A) = \{0\}$ independent.

<u>note</u> row reduction shows. Ⓐ has <u>dependent</u> rows $(0\,0\,0\,0)$

Ⓑ has <u>independent</u> rows.

<u>Summary</u> The $r = \text{rank}(A)$ non-zero rows of U and R are independent

The $r = \text{rank}(A)$ columns with pivots of U and R are independent.

<u>Example</u> I indep. C&P].

<u>Observation</u> A $n \times m$ matrix with $n > m$

row reduce

#pivots $\leq m < n$ so free variables

so columns are <u>dependent</u>.

<u>Corollary</u> a set of n vectors in $\mathbb{R}^m$ with $n > m$ <u>must</u> be dependent.

<u>Example</u> $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Spanning a vector space

V vector space. $w_1, \dots, w_\ell$ vectors in V

Defn We say $w_1, \dots, w_\ell$ span V if every vector in V is a linear combination $c_1 w_1 + c_2 w_2 + \dots + c_\ell w_\ell$ of $w_1, \dots, w_\ell$.

Remark this linear combination does not need to be unique.

Example
$$\begin{aligned} w_1 &= [1,0,0] \\ w_2 &= [0,1,0] \\ w_3 &= [-2,0,0] \end{aligned}$$
Span the xy-plane in $\mathbb{R}^3$.

Example The column space $C(A)$ is spanned by the columns.

Example The vector space V is spanned by all vectors in V.

Example The standard coordinate vectors span $\mathbb{R}^n$
$$\begin{aligned} e_1 &= [1,0, \dots 0] \\ e_2 &= [0,1, \dots 0] \\ &\vdots \\ e_n &= [0,0, \dots 1]. \end{aligned}$$

Example if $w_1, \dots, w_\ell$ span V, then adding any collection of vectors to $w_1 \dots, w_\ell$ also spans V.

special spanning sets: the minimal ones, i.e. no extra vectors.

Defn A basis for V is a collection of vectors which are

① linearly independent

② span V

Observation · every vector in V is a linear combination of basis vectors $c_1 b_1 + \dots + c_k b_k$. ($b_1 \dots b_k$ basis)

· this linear combination is unique

spose $v = a_1 b_1 + a_2 b_2 + \dots + a_n b_n = c_1 b_1 + c_2 b_2 + \dots + c_k b_k$.

then $v - v = a_1 b_1 + \dots + a_n b_n - c_1 b_1 - \dots - c_k b_k$.

$0 = (a_1 - c_1) b_1 + (a_2 - c_2) b_2 + \dots + (a_1 - c_k) b_k$

but b_i are linearly independent so only way to get zero vector is ~~through~~ if
$$\begin{aligned} a_1 - c_1 &= 0 \\ a_2 - c_2 &= 0 \\ &\vdots \\ a_k - c_k &= 0 \end{aligned}$$

i.e. $a_1 = c_1$, $a_2 = c_2$ .... $a_k = c_k$.