Factorization even if $A$ is not square, we can still record the row operations in a matrix, so if we do $A \rightsquigarrow U$ row-echelon form

not $R$ - reduced row-echelon form!

then $\underset{m\times n}{A} = \underset{m\times m}{L}\,\underset{m\times n}{U}$

$L$: lower triangular; $U$: upper triangular as before

Example $L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}$.

Solving $Ax=b$, $Ux=c$, $Rx=d$

case $b=0$ is special, because row operations don't change zeros.

(i.e. we don't write $\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & 0 \\ 2 & 6 & 9 & 7 & 0 \\ -1 & -3 & 3 & 4 & 0 \end{array}\right]$ ).

naive method (works for $Ax=b$, only need to solve for 1 value of $b$)

just do row operation on: $\left[\begin{array}{cccc|c} 1 & 3 & 3 & 2 & b_1 \\ 2 & 6 & 9 & 7 & b_2 \\ -1 & -3 & 3 & 4 & b_3 \end{array}\right]$

better method (when you need to solve for many different b's)

$Ax=b$. use $A=LU$ $LUx=b$.

$LUx=b$. recall $L$ invertible!

$Ux = L^{-1}x$

Example

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ 2b_1+b_2 \\ -b_1+2b_2+b_3 \end{bmatrix}$$

Q: is there a solution? A: iff $-b_1+2b_2+b_3=0$!

Observation $Ax=b$ can be solved iff $b$ lies in the column space of $A$ (not $U$!!)

even though $A$ has four columns, the column space is 2-dimensional, as the columns without pivots are sums of the other columns. $[C2 = 3C1,\ C4 = C3 - C1]$.

two descriptions of the column space $C(A)$: all linear combinations of $C_1, \cancel{C_2}, C_3, \cancel{C_4}$

: all solutions to $-b_1+2b_2+b_3=0$.

**Example** $b \in C(A)$, solve $Ax=b$ $\quad b=(1,5,5)$

$$Ax=b \qquad L=\begin{bmatrix}1&0&0\\2&1&0\\-1&2&1\end{bmatrix} \qquad U=\begin{bmatrix}1&3&3&2\\0&0&3&3\\0&0&0&0\end{bmatrix}.$$

$$L\underbrace{Ux}_{c}=b$$

solve: $Lc=b$: $\begin{bmatrix}1&0&0\\2&1&0\\-1&2&1\end{bmatrix}\begin{bmatrix}u\\v\\w\end{bmatrix}=\begin{bmatrix}1\\5\\5\end{bmatrix}$

$u=1$

$2u+v=5. \quad v=3.$

$-u+2v+w=5.$

$-1+6+w=5. \; w=0.$

solve $Ux=c$:

$$\begin{array}{c} \begin{array}{cccc} x & y & z & w \end{array} \\ \begin{bmatrix}\boxed{1}&3&3&2\\0&0&\boxed{3}&3\\0&0&0&0\end{bmatrix} \end{array}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}1\\3\\0\end{bmatrix}$$

$y=t. \qquad w=s.$

$3z+3w=3.$

$z=-s+1$

~~$x+3t-3s+2s=1.$~~

~~$x=1-3t+s$~~

$x+3t+3(-s+1)+2s=1.$

$x=-2-3t+s$

$$\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}-2-3t+s\\t\\-s+1\\s\end{bmatrix}=\underbrace{\begin{bmatrix}-2\\0\\1\\0\end{bmatrix}}_{x_p}+\underbrace{s\begin{bmatrix}1\\0\\-1\\1\end{bmatrix}+t\begin{bmatrix}-3\\1\\0\\0\end{bmatrix}}_{x_n}.$$

<u>key observation ①</u>

the solutions are all of the form $x_p + x_n$

$x_p$: some particular solution; $x_n$: solutions to $Ax=0$.

[note: if $x_p$ and $x_p'$ are two different solutions then $A(x_p-x_p')=\underbrace{Ax_p-Ax_p'}_{b-b}=0$.
so doesn't matter which $x_p$ you choose.]

<u>Reduced equations</u>

$$\left[\begin{array}{cccc|c}1&3&3&2&1\\0&0&3&3&3\\0&0&0&0&0\end{array}\right] \rightsquigarrow \begin{array}{c}\begin{array}{cccc} x & y & z & w\end{array}\\ \left[\begin{array}{cccc|c}\boxed{1}&3&\cancel{0}\,-1&-2\\0&0&\boxed{1}&1&1\\0&0&0&0&0\end{array}\right]\end{array}$$

$x=-2-3t+3s \qquad y$: free $t \qquad z=1-s. \qquad w$: free $s$

<u>key observation ②</u> $\underset{m\times n}{A}x=b.$

If there are <u>$r$ pivots</u> then there are <u>$n-r$ free variables</u>

**Defn** the number of pivots is the rank of A

Further more, the last $m-r$ rows of $U$ (and $R$) are zero!

**Summary** $Ax=b$ row reduces to $Ux=c$ and $Rx=d$.

with $r$ pivots. ($\text{Rank}(A)=r$) Then:

- the last $m-r$ rows of $U$ and $R$ are zero, so there is a solution iff the last $m-r$ entries of $c$ and $d$ are zero.
- the complete solution is $x=x_p+x_n$
  - can choose $x_p$ to have all free variables zero
  - $x_n$ are ~~all of~~ the solutions to $Ax=0$. } there are $(n-r)$-dimensional space of these.

**Example** $\left[\begin{array}{cccc:c} 1 & 2 & 3 & 5 & \\ 2 & 4 & 8 & 12 & \\ 3 & 6 & 7 & 13 & \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc:c} 1 & 2 & 3 & 5 & \\ 0 & 0 & 2 & 2 & \\ 0 & 0 & -2 & -2 & \end{array}\right] \rightsquigarrow \left[\begin{array}{cccc:c} 1 & 2 & 3 & 5 & \\ 0 & 0 & 2 & 2 & \\ 0 & 0 & 0 & 0 & \end{array}\right]$

## §2.3 Linear independence, basis, dimension

$\underset{m\times n}{A}$ matrix

$C(A)$ column space (# pivots $= r =$ rank)
$N(A)$ null space ($= m-r$)
} vector spaces! ← a vector space has a "size" called the dimension.

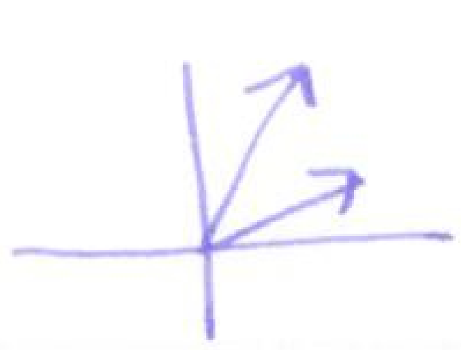
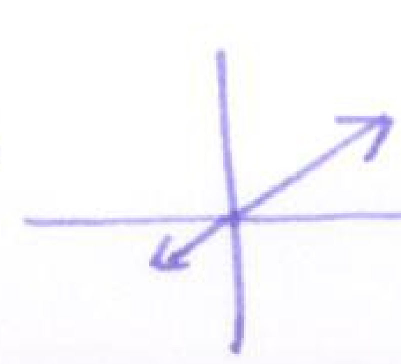
**Recall**: a linear combination of the vectors $v_1, \dots, v_k$ is a vector of the form $c_1v_1+c_2v_2+\dots+c_kv_k$.

**Defn** A set of vectors $v_1,\dots,v_k$ is linearly independent if $c_1v_1+c_2v_2+\dots+c_kv_k=0$ implies $c_1=0, c_2=0, \dots, c_k=0$. "the only way to get zero is to have every coefficient equal to zero".

If there is any non-trivial way to get zero, we say the vectors are linearly dependent i.e. $\exists\, c_1,\dots c_k$ not all zero such that $c_1v_1+c_2v_2+\dots+c_kv_k=0$.

**Observation** A set containing the zero vector is always linearly dependent.

If $v_1=0$ then choose $c_1=1, c_2=0, \dots c_k=0$ and $c_1v_1+\dots+c_kv_k=0$.

**Examples.**  independent.  dependent.  dependendent.