§2.2 Solving $Ax=0$ and $Ax=b$

so far: special case $Ax=b$ A $n\times n$ full set of pivots, unique solution $x=A^{-1}b$.

observations ① consider $Ax=b$ and suppose x_p is a solution $Ax_p=b$

x_n is in the null space, i.e. $Ax_n=0$.

then x_p+x_n is a solution.

check: $A(x_p+x_n) = \underbrace{Ax_p}_{b} + \underbrace{Ax_n}_{0} = b.$

② $0x=b$ has no solutions, unless $b=0$.

Examples

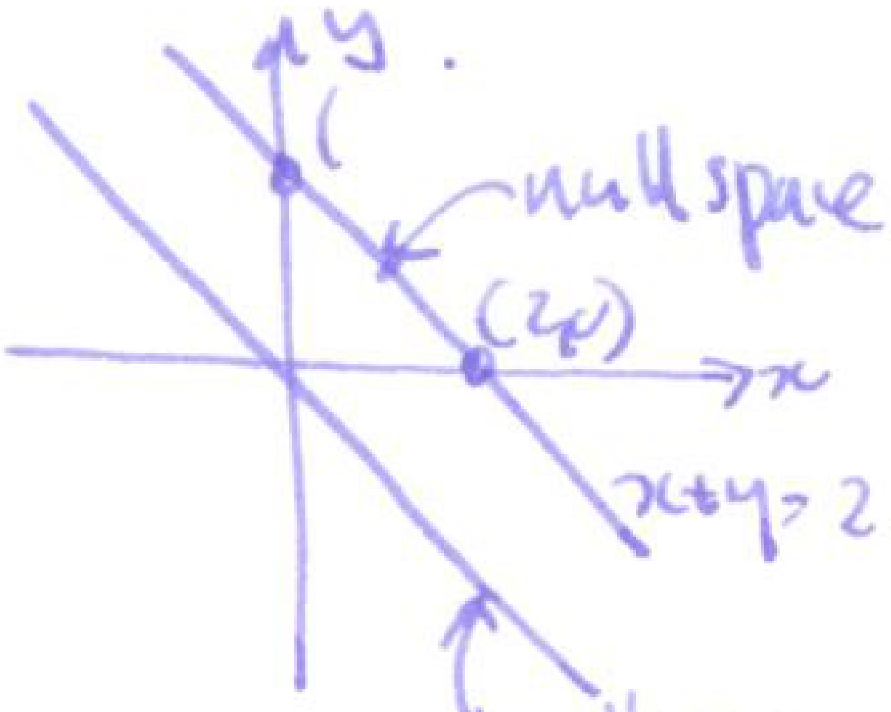
$$x+y=2 \qquad\qquad x+y=2$$
$$2x+2y=4. \qquad\qquad 2x+2y=5.$$

$$\left[\begin{array}{cc:c} 1 & 1 & 2 \\ 2 & 2 & 4 \end{array}\right] \qquad\qquad \left[\begin{array}{cc:c} 1 & 1 & 2 \\ 2 & 2 & 5 \end{array}\right].$$

$$\left[\begin{array}{cc:c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right] \qquad\qquad \left[\begin{array}{cc:c} 1 & 1 & 2 \\ 0 & 0 & 1 \end{array}\right] \leftarrow \text{no solutions!}$$

$y=s$

$x+s=2$ $\quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2-s \\ s \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$



null space + x_p particular soln

null space, line through 0, vector space.

Example

$$\begin{bmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \overset{\text{pivot}}{\boxed{1}} & 3 & 3 & 2 \\ 0 & 0 & \overset{\text{pivot}}{\boxed{3}} & 3 \\ 0 & 0 & 6 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

We can always row reduce to put the matrix in row-echelon form

① the pivots are the first non-zero entries in their rows.

② below each pivot is a column of zeros.

③ each pivot lies to the right of the pivot above.

we can go ~~one~~ two step further to get reduced row-echelon form

④ make all pivots 1 by dividing rows by pivots,

⑤ row reduce upwards to get zeros above the pivots.

$$\begin{bmatrix} 1 & 3 & 0 & \overset{-1}{\cancel{4}} \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Note solutions to $Ax=0$ are same as solutions to $Rx=0$, solve by backsubstitution

$$\begin{bmatrix} \overset{\text{pivot}}{\boxed{1}} & 3 & 0 & \overset{-1}{\cancel{4}} \\ 0 & 0 & \underset{\text{pivot}}{\boxed{1}} & 1 \\ 0 & \underset{\substack{\uparrow \\ \text{free} \\ \text{var}}}{0} & 0 & \underset{\substack{\uparrow \\ \text{free} \\ \text{var.}}}{0} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \overset{4}{} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

each column without a pivot corresponds to a free variable

$w = s$

$z + w = 0 \Rightarrow z = -s.$

$y = t$

$x + 3y \underset{-w}{\cancel{+4w}} = 0 \Rightarrow x = -3t \cancel{4w} + s.$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -3t \overset{+s}{\cancel{4s}} \\ t \\ -s \\ s \end{bmatrix} = \underbrace{t \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} \overset{1}{\cancel{4}} \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{this describes the null space. (2-dimensional)}}$$

Fact 2.2 2C If $Ax=0$ has more unknowns than equations $(m>n)$ then it has at least one free variable, i.e there is at least one non-trivial $(\neq 0)$ solution.

Preview # pivots = dimension of column space

\# free variables = dimension of null space