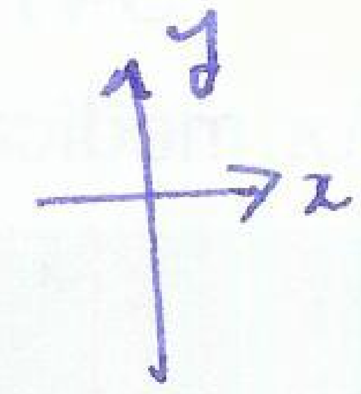
§2.1 Vector spaces and subspaces

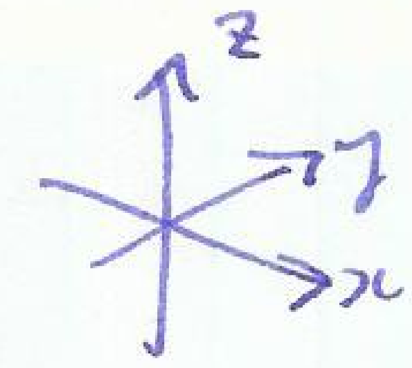
Examples $\mathbb{R}^n$

$\mathbb{R}$: numbers.

$\mathbb{R}^2$: $\langle x, y \rangle$.



$\mathbb{R}^3$. $\langle x, y, z \rangle$.



$\mathbb{R}^n$: $\langle x_1, x_2, \dots, x_n \rangle$

key property: linear combinations, i.e. we can add vectors, and takes scalar multiples.

Properties V vector space $x, y, z \in V$ $c_1, c_2 \in \mathbb{R}$.

addition: $x+y = y+x$ commutes

$x+(y+z) = (x+y)+z$ associative

$0+x = x+0 = x$ zero vector

$x+(-x) = 0$ inverse.

scalar mult: $1x = x$ identity

$(c_1c_2)x = c_1(c_2x)$

$c(x+y) = cx+cy$ dist.

$(c_1+c_2)y = c_1y+c_2$. dist.

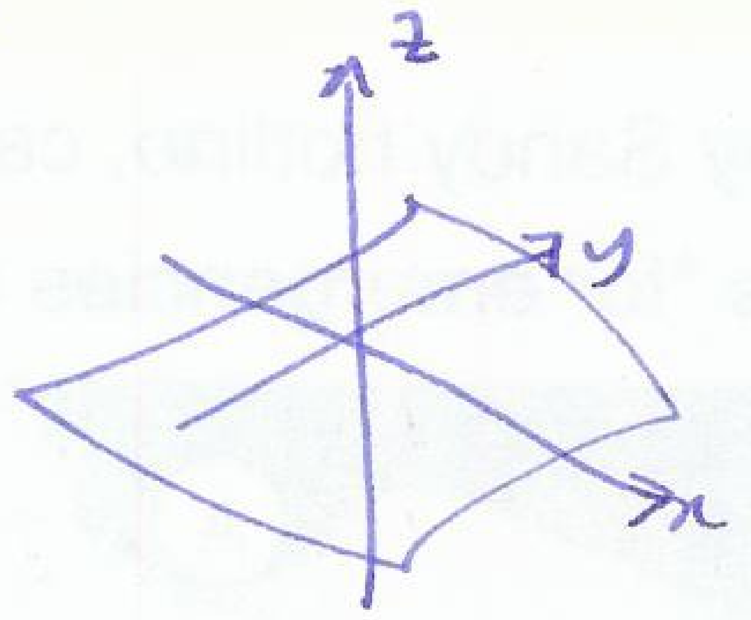
Examples • $\mathbb{R}^\infty$ $(x_1, x_2, x_3, \dots)$ $(0,0,0,\dots)$.
$(1,1,1,\dots)$.

• collection of 3×2 matrices. $\begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$ $A+B = B+A$. $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$.

• space of functions $f(x)$ on $[0,1]$ $f(x)+g(x)$. $cf(x)$. zero: $f(x)=0$.

subspaces.

Example



xy-plane $\subset \mathbb{R}^3$.

Def^{n} A subspace of a vector space is a non empty subset that satisfies the requirements for a subspace: Linear combinations stay in the space.

i.e. if x, y in subspace then $x+y$ in subspace
x in subspace then cx in subspace

Notation we say the subspace is closed under addition and scalar multiplication.

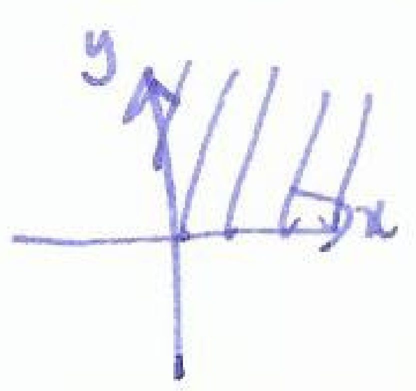
Observation • $\underset{\in \mathbb{R}}{0} \cdot x = 0$ vector, so every subspace must contain the zero vector.

• the smallest subspace is $\{0\}$ zero vector.

• the largest subspace of a vector space is all of V.

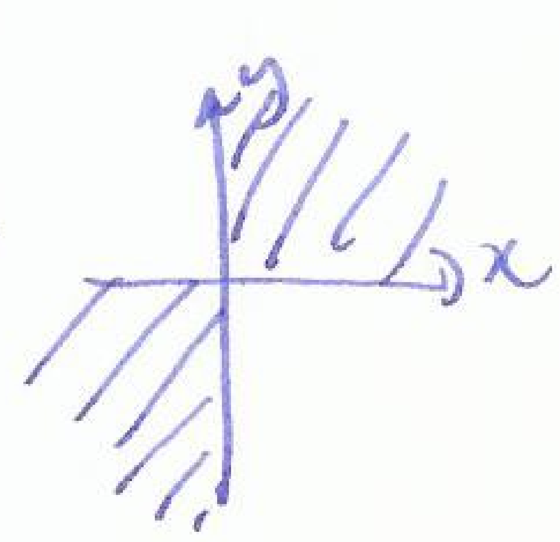
Examples

① all vectors in $\mathbb{R}^2$ with non-negative components $[x \; y]$ $x \geq 0, y \geq 0$



this is not a subspace. (satisfies addition but not scalar mult).

② 1st and 3rd quadrants.



(satisfies scalar mult but not addition)

③ $V =$ 3x3 matrices. subspaces: lower triangular matrices.
symmetric matrices.

Column spaces

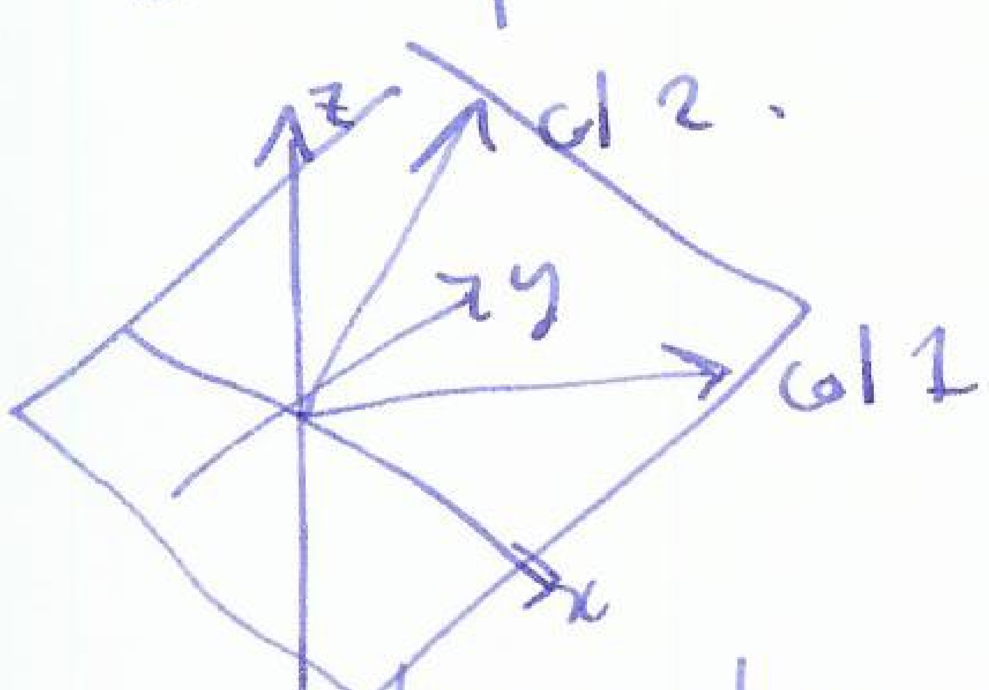
Let A be a matrix. The column space of A consists of all linear combinations of the columns of A, written $C(A)$.

Example
$$A = \begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}.$$ linear combination of cols: $c_1\begin{bmatrix}1\\5\\2\end{bmatrix} + c_2\begin{bmatrix}0\\4\\4\end{bmatrix}$.

e.g. $$\begin{bmatrix}2\\10\\4\end{bmatrix} + \begin{bmatrix}-0\\-4\\-4\end{bmatrix} = \begin{bmatrix}2\\6\\0\end{bmatrix}$$

~~Claim $C(A)$ is a subspace of $\mathbb{R}^3$.~~

Observation the linear system $Ax=b$ has a solution iff the vector b can be expressed as a linear combination of the cols of A, i.e. b is in the column space.



plane = column space.

Notation A $m \times n$ matrix then $C(A)$ is the column space consisting of all linear combinations of the columns.

Claim $C(A)$ is a subspace. $A = [a_1 \; a_2 \cdots a_n]$

Proof spose $b_1 = c_1 a_1 + c_2 a_2 + \cdots + c_n a_n$
$b_2 = d_1 a_1 + d_2 a_2 + \cdots d_n a_n$ then $b_1 + b_2 = (c_1+d_1)a_1 + (c_2+d_2)a_2 + \cdots + (c_n+d_n)a_n$.

alternatively spose $Ac = b_1$, $Ad = b_2$ then $A(c+d) = Ac + Ad = b_1 + b_2$.

If $b = c_1 a_1 + \cdots + c_n a_n$ then $cb = cc_1 a_1 + \cdots + cc_n a_n$ or $cAx = cb$.
$\Downarrow$
$A(cx) = cb$. $\square$

Example A 5×5 matrix. how big can $C(A)$ be?

$A = 0$ then $C(A) = \{0\} \subset \mathbb{R}^5$.

$A = I$ then $C(A) = \mathbb{R}^5$.

Observation if A is $n\times n$ and non-singular then $C(A) = \mathbb{R}^n$

because $Ax = b$ has a (unique) solution for every $b \in \mathbb{R}^5$.

Nullspaces

Defn The nullspace of a matrix A consists of all ~~real~~ vectors x such that $Ax = 0$ if A is $m\times n$ then nullspace of $\mathbb{R}^m$

Claim the nullspace of A is a vector space.

Proof addition: if $Ax = 0$ and $Ay = 0$ then $A(x+y) = Ax + Ay = 0 + 0 = 0$

scalar multiplication: if $Ax = 0$ then $A(cx) = c(Ax) = c0 = 0$.

Warning the solutions to $Ax = b$ ($b \neq$ zero vector) does not form a vector space!

Example ①
$$\begin{bmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

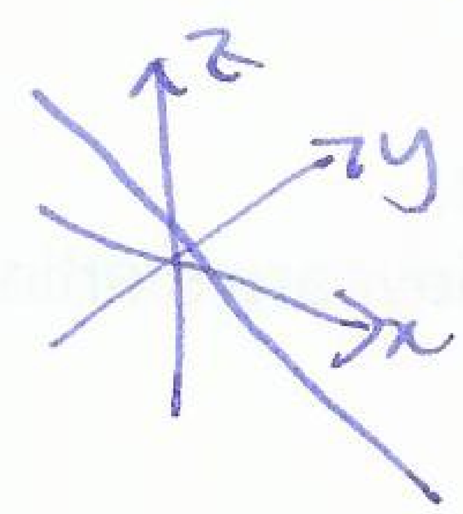
$u = 0$

$5u + 4v = 0 \Rightarrow v = 0$.

so nullspace $= \{[0, 0]\}$.

②
$$\begin{bmatrix} 1 & 0 & 1 \\ 5 & 4 & 9 \\ 2 & 4 & 6 \end{bmatrix}\begin{bmatrix} c \\ c \\ -c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so nullspace is any multiple of $\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$.



i.e. a line of solutions.