- E_{ij} elementary matrix, ~~add~~ subtract a multiple l of row j from row i.
- P_{ij} swap rows i,j.
- D (or D^{-1}) divide all rows by their pivots.

so

$$\underbrace{(D^{-1} \cdots E_i \cdots P \cdots E_2 E_1)}_{\text{left inverse!}} A = I$$

must equal A^{-1} by ②: ~~AB~~ $(BA)C = B(AC)$.

$\Rightarrow$ suppose A does not have n pivots. then can row reduce to $\begin{bmatrix} d_1 & * & * & * \\ 0 & d_2 & * & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \end{bmatrix}$

so when you reduce upwards, get whole row of zeros, so there is non-zero x s.t. $Ax = 0$. □

The transpose matrix

A ~~$n \times m$~~ $m \times n$ matrix A^T is an $n \times m$ matrix with $(A^T)_{ij} = A_{ji}$.

Example. $A = \begin{bmatrix} 2 & 1 & 4 \\ 0 & 0 & 3 \end{bmatrix}$ $A^T = \begin{bmatrix} 2 & 0 \\ 1 & 0 \\ 4 & 3 \end{bmatrix}$.

useful facts:
- $(A^T)^T = A$
- $(\text{upper triangular})^T = (\text{lower triangular})$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A^{-1})^T = (A^T)^{-1}$ ← check

$$A A^{-1} = I.$$
$$(AA^{-1})^T = I^T = I$$
$$(A^{-1})^T A^T = I \Rightarrow (A^{-1})^T \text{ is an inverse for } A^T \text{, i.e. } (A^T)^{-1}$$

Symmetric matrices

A matrix A is <u>symmetric</u> if $A^T = A$.

Examples I. $[0]$ $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{bmatrix}$

Fact if A is symmetric, and A^{-1} exists, then A^{-1} is symmetric.

Proof spse. $A^T = A$. $\underbrace{(A^T)^{-1}}_{A^{-1}} = (A^{-1})^T \Rightarrow A^{-1} = (A^{-1})^T \Rightarrow A$ symmetric $\square$.

Symmetric product

Fact Let R be any matrix ($m \times n$) then $R^T R$ is a square symmetric matrix.

Proof $\underset{n\times m}{R^T}\,\underset{m\times n}{R} = n \times n$ $\quad (R^T R)^T = R^T (R^T)^T = R^T R. \Rightarrow$ symmetric. $\square$.

Example $\begin{bmatrix} 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = [5]$. $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Fact suppose A is symmetric ($A = A^T$) and $A = LDU$ with no row swaps then $U = L^T$, so $A = LDL^T$. This is the symmetric factorization.

Proof $A = LDU$
$A^T = U^T D^T L^T = U^T D L^T$ but U, L unique $\Rightarrow L = U^T$. $\square$.

§1.7 Special matrices and applications

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Example solve $-\frac{d^2u}{dx^2} = f(x)$ (*) $\quad 0 \le x \le 1$

solution: a function $u(x)$.

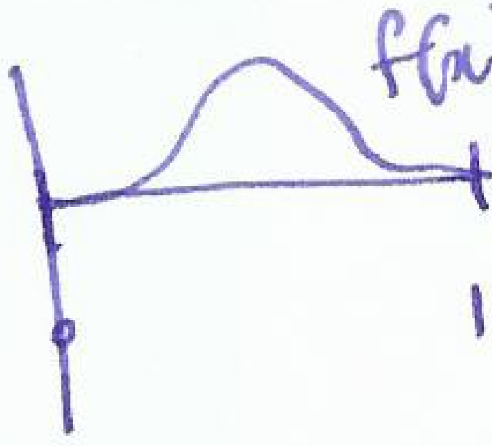
note if $u(x)$ is a solution $\quad u(x) + C + Dx$ is also a solution.

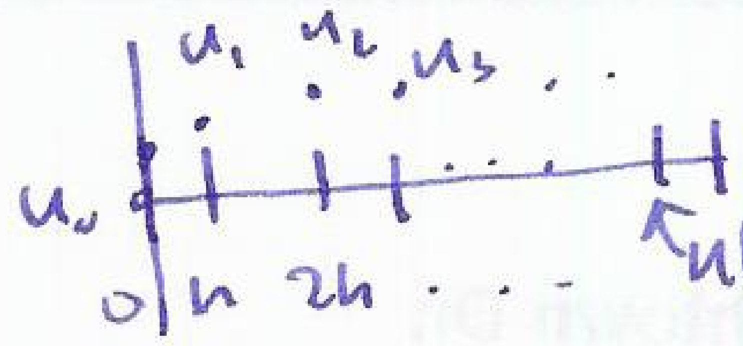
e.g. $f(x) = 0$. then $u(x) = C + Dx$ is sol$^{\underline{n}}$.

boundary conditions: $\quad u(0) = 0 \quad u(1) = 0 \quad \rightsquigarrow$ determines C, D.

application: temperature distribution ~~heat flow~~ in a rod with heat source $f(x)$.

aim: find approximate solution by discretization.



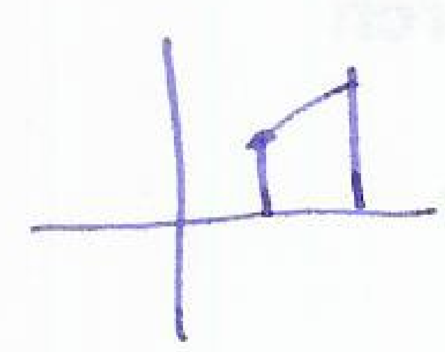


set $u_i \approx f(ih)$.

$x = 0$: $u_0 = 0$

$x = 1 = (n+1)h \quad u_{n+1} = 0$.

approximate derivative $\quad \frac{\Delta u}{\Delta x} = \frac{u(x+h) - u(x)}{h}$.



symmetric version: $\frac{du}{dx} \approx \frac{\Delta u}{\Delta x} = \frac{u(x+h) - u(x-h)}{2h}$

2nd derivative: $\frac{d^2u}{dx^2} \approx \frac{\Delta^2 u}{\Delta x^2} = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2}$

(*): $\quad \frac{-u_{j+1} + 2u_j - u_{j-1}}{h^2} = f(jh)$

Example $h = \frac{1}{6}$

$$A = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f(h) \\ f(2h) \\ f(3h) \\ f(4h) \\ f(5h) \end{bmatrix}$$

properties:
- tridiagonal (sparse)
- symmetric $A = A^T$ $\quad A = LDL^T$
- positive definite $\Rightarrow$ all pivots are positive.

Elimination:

$(2) - \frac{1}{2}(1)$
$$\begin{bmatrix} 2 & -1 & & & \\ 0 & 3/2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & 1 & 2 \end{bmatrix} \cdots$$

note: · only one row operation
· pivot row short.

obtain

$$A = \begin{bmatrix} 1 & & & & \\ -1/2 & 1 & & & \\ & -2/3 & 1 & & \\ & & -\frac{3}{4} & 1 & \\ & & & -\frac{4}{5} & 1 \end{bmatrix} \begin{bmatrix} 2/1 & & & & \\ & 3/2 & & & \\ & & 4/3 & & \\ & & & 5/4 & \\ & & & & 6/5 \end{bmatrix} \begin{bmatrix} 1 & -1/2 & & & \\ & 1 & -2/3 & & \\ & & 1 & -3/4 & \\ & & & 1 & -4/5 \\ & & & & 1 \end{bmatrix}$$

L (bidiagonal) $\qquad D \qquad L^T$

$\det(A) = \det(D)$

$= \frac{\cancel{2}}{1} \cdot \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{4}}{\cancel{3}} \cdot \frac{\cancel{5}}{\cancel{4}} \cdot \frac{6}{\cancel{5}} = 6.$

complexity: number of operations $\sim n$

Roundoff error

Example

$A = \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0.0001 & 1 \\ 1 & 1 \end{bmatrix}$

Ill conditioned. $\quad$ well-conditioned.

$u + v = 2$ $\qquad$ $u + v = 2$

$u + 1.0001v = 2$ $\qquad$ $u + 1.0001v = 2.0001$

solution $u = 2$, $v = 0$ $\qquad$ $u = 1$, $v = 1$

naive elimination on B :

$$\left[\begin{array}{cc|c} 0.0001 & 1 & 1 \\ 1 & 1 & 2 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 0.0001 & 1 & 1 \\ 0 & -9999 & -9998 \end{array}\right]$$

round off in back substitution, get $v = 1$, $u = 0$

correct result : $v = 0.9999$ $\quad u = 1$.

moral: small pivots bad!

solution : for best numerical results exchange rows to get largest pivot.

this is called : elimination with partial pivoting.