symmetric version: $A = LDU$

L: lower triangular, 1s on diagonal
D: diagonal matrix.
U: upper triangular, 1s on diagonal

Example

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Fact If $A = LU$ then L, U are unique.
$= LDU$ LDU are unique.

General case: pivots are zero.

Example $\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ swap rows: $\begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} b_2 \\ b_1 \end{bmatrix}$.

Observation we can swap rows by a matrix, called a permutation matrix

in this case $P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ check $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 2 \end{bmatrix}$

original system $Ax = b$
new system $PAx = Pb$

Def^n A permutation matrix has the same rows as the identity in some order.
there is a single 1 in each row and column.

Example $P = I_n$! $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

Q. how many permutation matrices? **A** $n!$

Facts: the product of two permutation matrices is another permutation matrix.

Fact $P^{-1} = P^T$ where $(P^T)_{ij} = (P)_{ji}$ (turn P on its side).

Example

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Fact (1.5 1J) In the non-singular case (nxn, unique solution [n pivots]) there is a permutation matrix P that reorders the rows of A to avoid zeros in the pivots. Then $PAx = b$ has a unique solution and $PA = LU$ or LDU.

In the singular case (not enough pivots) is slightly more complicated.

§1.6 Inverses and transposes

Let A be an $n \times n$ matrix. The inverse of A is another $n \times n$ matrix, usually called A^{-1} such that $A^{-1}A = I_n$ and $AA^{-1} = I_n$. [~~$AA^{-1}A = I$~~ $AA^{-1}AA^{-1} = AIA^{-1} = AA^{-1}$]

warning: not all $n \times n$ matrices have an inverse.

Example. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ ← check $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & a \\ 0 & c \end{bmatrix}$

Observations

① The inverse exists iff elimination produces n pivots. (i.e. unique solution)

②a ~~If there is an inverse, it is unique.~~ left inverse = right inverse.

Proof suppose $BA = I$ & $AC = I$ then consider $(BA)C = B(AC)$

$IC = BI \Rightarrow C = B$.

~~②b Note this also shows that a left inverse $BAC = I$ is the same as the right inverse.~~

②b inverse is unique. spse $BA = I$, $CA = I$ then $(BA)C = B(AC)$

$IC = BI$ $C = B$. □.

③ If A is invertible then $Ax = b$ has the unique solution $x = A^{-1}b$.

$Ax = b$

$A^{-1}Ax = A^{-1}b$.

$Ix = A^{-1}b$

$x = A^{-1}b$.

(4) spose there is a non-zero vector x s.t. $Ax = 0$ then A <u>cannot</u> have an inverse as $A^{-1}Ax = A0 = 0 \Rightarrow x = A0 = 0$ #.

i.e. if A has an inverse, then $Ax = 0$ has exactly one solution $x = 0$.

(5) 2x2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible iff $\underbrace{ad-bc}_{=\det(A)} \neq 0$

if $ad - bc \neq 0$ then $A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

check! $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-cd & ad-bc \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

(6) a diagonal matrix has an inverse if no diagonal elements are zero.

$$\begin{bmatrix} d_1 & & 0 \\ & d_2 & \\ 0 & & \ddots \; d_n \end{bmatrix}^{-1} = \begin{bmatrix} 1/d_1 & & 0 \\ & 1/d_2 & \\ 0 & & \ddots \; 1/d_n \end{bmatrix}.$$

<u>Prop^n</u> (1.6.1L) If A has inverse A^{-1}, B " " B^{-1} then $\boxed{(AB)^{-1} = B^{-1}A^{-1}}$

warning: order changes!

check $ABB^{-1}A^{-1} = AIA^{-1} = AA^{-1} = I$.

<u>Corollary</u> $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

<u>Q</u>: How do we find A^{-1}? A: elimination/row operations!

<u>idea</u> $AA^{-1} = I$ think of this as an equation and solve for A.

$$A\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} Ax_1 & Ax_2 & Ax_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

could solve: $Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $Ax_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ $Ax_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. separately

better: do all three at the same time!

example $A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}$ solve $Ax_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = e_1$ $\begin{bmatrix} 2 & 1 & 1 & 1 \\ 4 & -6 & 0 & 0 \\ -2 & 7 & 2 & 0 \end{bmatrix}$.

do elimination / row reduction on: $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 4 & -6 & 0 & 0 & 1 & 0 \\ -2 & 7 & 2 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} A & I \end{bmatrix}$.

row operations: $\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 8 & 3 & 1 & 0 & 1 \end{bmatrix}$.

$$\begin{bmatrix} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -8 & -2 & -2 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} U & L^{-1} \end{bmatrix}$$

now clear entries <u>above</u> the pivots.

$$\begin{bmatrix} 2 & 1 & 0 & 2 & -1 & -1 \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 & 0 & \frac{12}{8} & -\frac{5}{8} & -\frac{6}{8} \\ 0 & -8 & 0 & -4 & 3 & 2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix}$$

<u>Fact</u>: $\det(A)$ = product of pivots $= -16$.

divide by pivots: $\begin{bmatrix} 1 & 0 & 0 & 12/16 & -5/16 & -6/16 \\ 0 & 1 & 0 & 1/2 & -3/8 & -1/4 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}$ check!.

<u>Fact</u> this method of computing A^{-1} takes $\leq \frac{4}{3}n^3$ operations. In fact n^3.

<u>Thm</u> A $n \times n$ matrix is invertible $\Leftrightarrow$ A has a full set of n pivots when you do row reduction.

<u>$\Leftarrow$</u> suppose A has a full set of pivots. Then solving $AA^{-1} = I$ equivalent to solving $Ax_i = e_i$, unique solution, $\Rightarrow A^{-1} = [x_1 \cdots x_n]$ is inverse.

~~$\Rightarrow$~~ <u>Fact</u> A 1-sided inverse of a square matrix is automatically a 2-sided inverse.

<u>Proof</u> consider row reductions $A \rightsquigarrow I$ every row reduction operation is one of: