§1.5 Triangular factors and row exchanges

back to elimination example:
$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 0 \\ -2 & 7 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix} \qquad Ax = b$$

$$\begin{array}{l} 1 \\ 2 - 2(1) \\ 3 \end{array} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}}_{EA}$$

$$\begin{array}{l} 1 \\ 2 \\ 3 + 1 \end{array} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{F} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}}_{EA} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}}_{FEA}$$

$$\begin{array}{l} 1 \\ 2 \\ 3 + 2 \end{array} \quad \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{G} \underbrace{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}}_{FEA} = \underbrace{\begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 0 & 1 & 2 \end{bmatrix}}_{GFEA}$$

upper triangular - can solve by back substitution.

$Ax = b$.

$$\begin{aligned} 2u + v + w &= 5 \qquad u = 1 \\ -8v - 2w &= -12 \qquad v = 1 \\ w &= 2. \end{aligned}$$

$$\underbrace{GFE}_{\text{elementary matrices}}\underbrace{A}_{\text{original matrix}} = \underbrace{U}_{\text{upper triangular.}}$$

key property (1): a product of upper triangular matrices is upper triangular
" lower " lower "

$\underbrace{GFE}$
lower triangular!

key property (2): each row operation is reversible so elementary matrix has an inverse.

Def$^{\underline{n}}$ an inverse for $\underset{(n\times n)}{A}$ is a matrix A^{-1} such that $A^{-1}A = I_n$ identity matrix.

warning: not all matrices have inverses! (e.g. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ zero matrix).

Example

$$\begin{array}{l} ① \\ ② - 2① \\ ③ \end{array} \qquad \begin{array}{l} ① \\ ② + 2① \\ ③ \end{array}$$

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \qquad \underset{E^{-1}}{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

check!

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \checkmark.$$

$$GFEA = U$$

$$\underbrace{G^{-1}G}_{I}FEA = G^{-1}U$$

$$\underbrace{F^{-1}F}_{I}EA = F^{-1}G^{-1}U.$$

$$\underbrace{E^{-1}E}_{I}A = E^{-1}F^{-1}G^{-1}U.$$

$$A = \underbrace{E^{-1}F^{-1}G^{-1}}_{\text{lower triangular}}\ \underbrace{U}_{\text{upper triangular}}$$

$$A = \underbrace{L}_{\substack{\text{lower triangular} \\ \text{1's on diagonals}}}\ \underbrace{U}_{\substack{\text{upper triangular (result of doing Gaussian} \\ \text{elimination on } A).}} \qquad \text{"LU factorization."}$$

<u>Thm</u> Every matrix A has an LU factorization, $A = LU$ where

U is upper triangular, result of doing Gaussian elimination.

L is lower triangular, 1's on the diagonal, corresponds to row operations.

~~Also~~

Thm (1.5 1H) If A can be row reduced with no exchanges of rows, then $A = LU$ where L is lower triangular, with 1s on diagonal. U upper triangular

Q: why did we do this?

A: spose we want to solve $\underbrace{Ax = b}_{\substack{\text{linear system} \\ (n^3)}} \leftrightarrow \underbrace{LUx = b}$

↑ two triangular systems, i.e.

solve $Lc = b$. (n^2).

then $Ux = c$ (n^2)

we often want to solve $Ax = b$ for many diffent values of b.

so we can do this by ① factor $A = LU$ ($\frac{1}{3}n^3$ operations)

② solve $\begin{matrix} Lc = b \\ Ux = c \end{matrix}$ ($2n^2$ operations).

Example solve $Ax = b$

$$\begin{aligned} x + 2y - z &= b_1 \\ -x \quad\;\; + 3z &= b_2 \\ x + 6y + 6z &= b_3. \end{aligned} \qquad A = \begin{bmatrix} 1 & 2 & -1 \\ -1 & 0 & 3 \\ 1 & 6 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ +1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 4 & 7 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$b = \begin{bmatrix} +1 \\ 1 \\ -1 \end{bmatrix}$ solve $\begin{matrix} Lc = b \\ Ux = c \end{matrix}$

$$Lc = b: \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$c_1 = 1$

$-c_1 + c_2 = 1$.

$c_2 = 2$.

$c_1 + 2c_2 + c_3 = -1$.

$1 + 4 + c_3 = -1$

$c_3 = -6$

$$Ux = c \quad \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -6 \end{bmatrix}$$

$x + 2y - z = -6$ $\quad x = -14$

$2y + 2z = 2$ $\quad y = 3$

$3z = -6$ $\quad z = -2$.

check!