Example

row view.

$$\underset{2\times 3}{\overset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}}} \underset{3\times 2}{\overset{B.}{\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}}} = \overset{2\times 2}{\begin{bmatrix} (AB)_{11} & (AB)_{12} \\ (AB)_{21} & (AB)_{22} \end{bmatrix}}$$

$$(AB)_{11} = (\text{1st row of } A)\ \underset{\text{dot}}{}\ (\text{1st column of } B) = a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} = \sum_{j=1}^{n} a_{1j} b_{j1}$$

$$(AB)_{ij} = (i\text{th row of } A) \cdot (j\text{th column of } B).$$

$$= \sum_{k=1}^{n} a_{ik} b_{kj}$$

column view

$\underset{m\times n}{A}\ \underset{n\times p}{B}$ $\quad$ B has columns $\begin{bmatrix} b_1 & b_2 & \cdots & b_p \end{bmatrix}$

$$AB = \begin{bmatrix} Ab_1 & Ab_2 & \cdots & Ab_p \end{bmatrix}$$

Q: why do we use this definition of matrix multiplication.

A: because it is useful.

Observation: Gaussian elimination operations are given by matrix multiplication by special matrices called elementary matrices

recall:

$$\begin{aligned} 2u + v + w &= 5 \quad ① \\ 4u - 6v \quad\quad &= -2 \quad ② \\ -2u + 7v + 2w &= 9 \quad ③ \end{aligned} \qquad \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix}.$$

step ①: $\left.\begin{matrix} ① \\ ② - 2① \\ ③ \end{matrix}\right\}$ $\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 4 & -6 & 0 & -2 \\ -2 & 7 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix}$

step ②: $\begin{matrix} ① \\ ② \\ ③ + ① \end{matrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ -2 & 7 & 2 & 9 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 & 5 \\ 0 & -8 & -2 & -12 \\ 0 & 8 & 3 & 14 \end{bmatrix}$

can also swap rows: $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ a_{41} & a_{42} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \\ a_{21} & a_{22} \\ a_{41} & a_{42} \end{bmatrix}$$

swaps middle two rows.

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

swaps rows 1 and 3.

<u>Properties of matrix multiplication</u>

$$\underset{m\times n}{A} \quad \underset{n\times p}{B} \qquad \underset{m\times p}{\underline{AB}}$$

- each entry $(AB)_{ij}$ of AB is the product of the ith row of A with the jth column of B
- ~~each~~ the j-th column of AB is the product of A by the j-th column of B.
- the ith row of AB is row i of A times B.

<u>Associativity</u> $(AB)C = A(BC)$ so can just write ABC.

<u>distributive</u> $A(B+C) = AB + AC$

$(A+B)C = AC + BC$

<u>not commutative!</u> $AB \neq BA$

<u>note</u>: $\underset{m\times n}{A} \; \underset{n\times p}{B}$ makes sense $\quad \underset{n\times p}{B} \; \underset{m\times n}{A}$ doesn't work!

but even if A, B $n\times n$ $AB \neq BA$ in general.

<u>Example</u>

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}$$