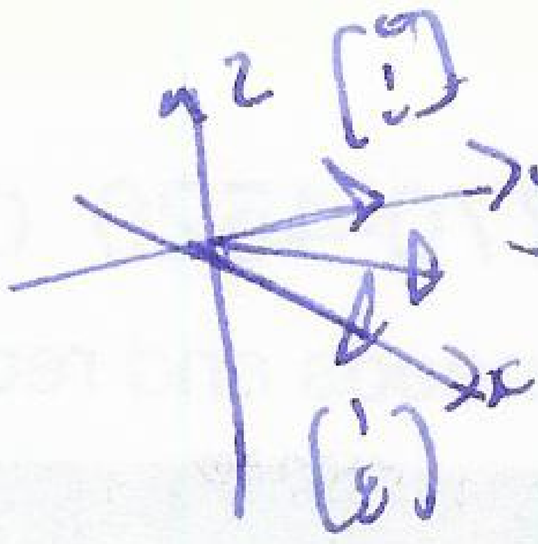
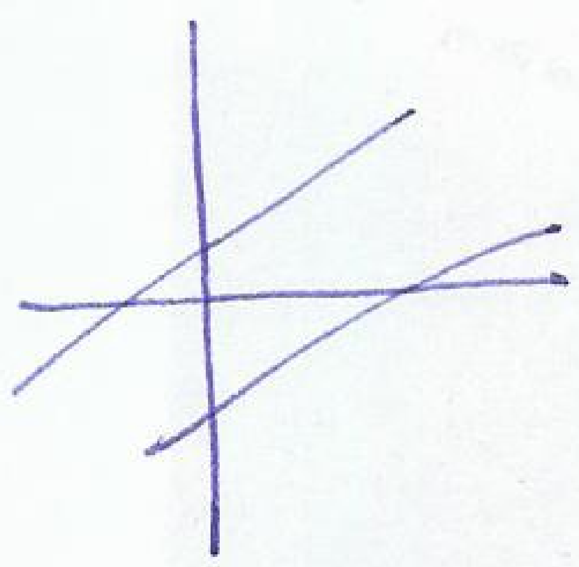
problem case: $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}$  $\begin{bmatrix}1\\1\\0\end{bmatrix}$ } all linear combination only give $\mathbb{R}^2 \subset \mathbb{R}^3$.

③ linear map. $Ax=b : \mathbb{R}^3 \to \mathbb{R}^3$ Q. is $b \in$ image(A)?

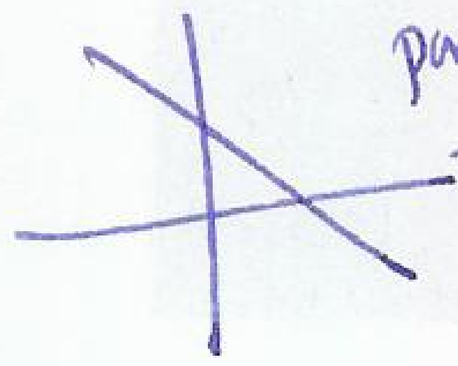
Q: what goes wrong in elimination?



parallel lines no solution

$x+y=1$ ① $x+y=1$

$x+y=2$. ② ②−① $0x+0y=+1$ } no solutions!



same parallel lines – infinitely many solutions

$x+y=1$ ① $x+y=1$

$2x+2y=2$ ② ②−2① $0x+0y=0$ } only one equation – a whole line of solutions.

§1.3 Gaussian elimination example

$2u+v+w=5$ ①

$4u-6v=-2$ ②

$-2u+7v+2w=9$ ③.

notation: pivot

① $\boxed{2}u+v+w=5$

②−2① $-8v-2w=-12$

③+① $8v+3w=14$

① $2u+v+w=5$

② $-8v-2w=-12$

③+② $1w=2$

} triangular system, can solve via back-substitution

$w=2$

$-8v-4=-12 \Rightarrow v=1$

$2u+1+2=5 \Rightarrow u=1$

better notation:

$$\begin{bmatrix}2&1&1&5\\4&-6&0&-2\\-2&7&2&9\end{bmatrix} \to \begin{bmatrix}2&1&1&5\\0&-8&-2&-12\\0&8&3&14\end{bmatrix} \mapsto \begin{bmatrix}\boxed{2}&1&1&5\\0&\boxed{-8}&-2&-12\\0&0&\boxed{1}&2\end{bmatrix}.$$

pivots.

Important: pivots cannot be zero.

Q: What can go wrong? A: a zero can appear in a pivot position

Example (avoidable) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 6 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 2 & 4 \end{bmatrix} \xrightarrow{\text{swap!}} \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & \boxed{2} & 4 \\ 0 & 0 & \boxed{3} \end{bmatrix}$

Example (unavoidable) $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 5 \\ 4 & 4 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \\ 0 & 0 & 4 \end{bmatrix} \to \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & \boxed{3} \\ 0 & 0 & 0 \end{bmatrix}$

only 2 pivots!

How many operations do we need for elimination?

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{n1} & & \cdots & a_{nn} \end{bmatrix} \rightsquigarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & & & \\ \vdots & & \begin{bmatrix} & \\ & \end{bmatrix} & \\ 0 & & & \end{bmatrix}$$

rows. $(n-1)\times n$ operations.

$(n-1)\times(n-1)$ matrix.

so need

$$(n-1)n + (n-2)(n-1) + \cdots \text{ operations}$$

$$= n^2 - n + (n-1)^2 - (n-1) + \cdots + 1^2 - 1$$

$$= \underbrace{1 + 2^2 + 3^2 + \cdots n^2}_{\frac{n(n+1)(2n+1)}{6}} - \underbrace{(1+2+3+\cdots+n)}_{\frac{n(n+1)}{2}} = \frac{n^3 - n}{3} \sim \frac{1}{3}n^3.$$

§1.4 Matrix notation and matrix multiplication

Example.

$$\begin{aligned} 2u + v + w &= 5 \\ 4u - 6v \quad &= -2 \\ -2u + 7v + 2w &= 9. \end{aligned}$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 4 & -6 & 6 \\ -2 & 7 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ 9 \end{bmatrix}.$$

9 coefficients - coefficient matrix A | 3 unknowns x | 3 right hand side numbers. b.

$Ax = b$.

Def^{n} A matrix is a rectangular array of numbers. $m \times n$ matrix, rows $\times$ columns.

~~operations~~ examples

$[3]$ — 1×1 $\quad$ $[2 \ 1]$ — 1×2 $\quad$ $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ — 3×1. $\quad$ $\begin{bmatrix} 2 & 3 & 4 \\ 1 & 1 & 1 \end{bmatrix}$ — 2×3.

operations

addition if two matrices have the same size, then we may add them.

Example

$$\begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 4 & 3 \end{bmatrix}.$$

$$[1 \ 0 \ 1] + [-1 \ -1 \ -1] = [0, -1, 0].$$

scalar multiplication you can multiply a matrix by a number.

Example

$$2\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}.$$

matrix multiplication

special case

$$\underset{1\times 3}{[1 \ 2 \ 3]} \underset{3 \times 1}{\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}} = [1.4 + 2.5 + 3.6] = \underset{1 \times 1}{[32]}.$$

$$4 + 10 + 18$$

General cases (2 equivalent views).

$$\underset{3\times 3}{\overset{A}{\begin{bmatrix} 1 & 1 & 6 \\ 3 & 0 & 1 \\ 1 & 1 & 4 \end{bmatrix}}} \underset{3\times 1}{\overset{x}{\begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix}}} = \begin{bmatrix} 1\cdot 2 + 1\cdot 5 + 6\cdot 0 \\ 3\cdot 2 + 0\cdot 5 + 1\cdot 0 \\ 1\cdot 2 + 1\cdot 5 + 4\cdot 0 \end{bmatrix}$$

inner product of each row of first matrix with each column of second matrix.

equivalently: Ax is $2\begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} + 5\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$

a linear combination of the columns of A

more generally: $\underbrace{A}_{m\times n}\underbrace{x}_{n\times 1}$ $\quad \underbrace{Ax}_{m\times 1}$

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & & & \\ \vdots & & & \\ a_{m1} & a_{m2} \cdot & & a_{mn} \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}$$

$$Ax = \begin{bmatrix} (Ax)_1 \\ (Ax)_2 \\ \vdots \\ (Ax)_m \end{bmatrix}$$

$$(Ax)_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = \sum_{j=1}^{n} a_{1j} x_j$$

$$(Ax)_{\cancel{j}\,i} = a_{j1}x_1 + a_{j2}x_2 + \cdots + a_{jn}x_n = \sum_{j=1}^{n} a_{ij} x_j$$

Special matrix: identity matrix $n\times n$ matrix $I_n = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$.

$I_1 = [1]$ $\quad I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\quad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ etc.

key property $Ix = x$

Matrix multiplication $\quad \underbrace{\underbrace{A}_{m\times n}\underbrace{B}_{n\times p}}_{m\times p.}$

$\underset{\text{rows}\times\text{cols}}{A}$ $\quad \underset{\text{rows}\times\text{cols}}{B}$

these must be equal!