Math 338 Linear Algebra

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– students with disabilities

Text: Linear algebra, Strang

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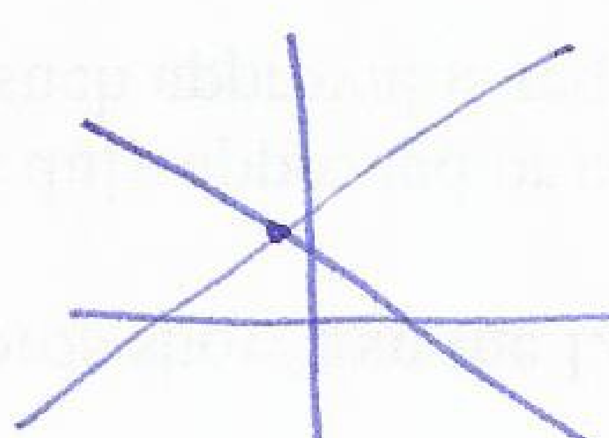
§1.1 Introduction

aims:
- solve linear equations
- understand linear maps/functions

Example 2 equations in 2 unknowns

$$1x + 2y = 3 \quad (1)$$
$$4x + 5y = 6 \quad (2)$$

expect unique solution.



two lines usually intersect in a unique point.

Elimination : (2) − 4(1) :

$$\begin{aligned} 4x + 5y &= 6 \\ -4x - 8y &= -12 \\ \hline -3y &= -6 \end{aligned} \qquad y = 2$$

substitute in (1) : $x + 4 = 3$

$x = -1$.

Determinants x, y functions of coeffs of (1), (2), so maybe there is a formula for x and y check!

$$y = \frac{\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}}{\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix}} = \frac{1 \cdot 6 - 3 \cdot 4}{1 \cdot 5 - 2 \cdot 4} = \frac{-6}{-3} = 2$$

$$x = \frac{\begin{vmatrix}3 & 2\\ 6 & 5\end{vmatrix}}{\begin{vmatrix}1 & 2\\ 4 & 5\end{vmatrix}} = \frac{3\cdot 5 - 2\cdot 6}{1\cdot 5 - 2\cdot 4} = \frac{3}{-3} = -1.$$

Q: what about 1000 equations in 1000 variables?

Elimination: efficient (n^3)

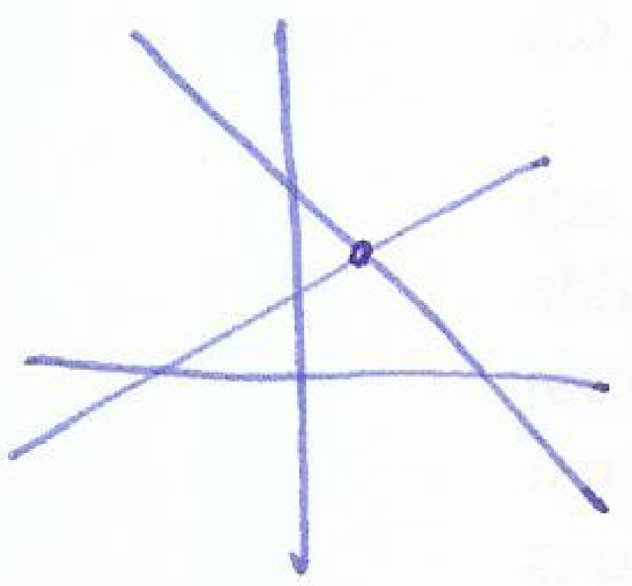
Determinants: inefficient ($n!$)

<u>Problem</u>: not all ~~sols~~ systems of linear equations have a unique solution.

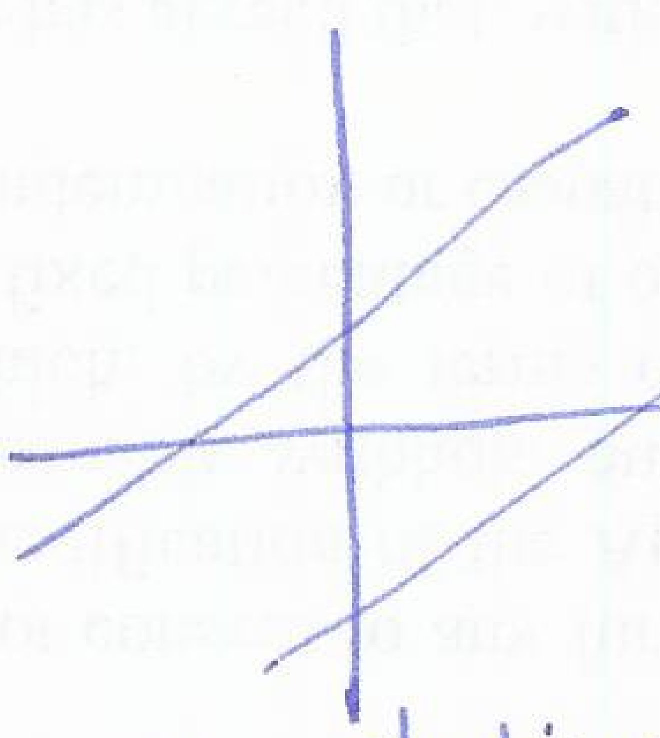
(even when there are the same number of vars and equations)

① We can understand what happens geometrically: 2 equations, 2 unknowns.

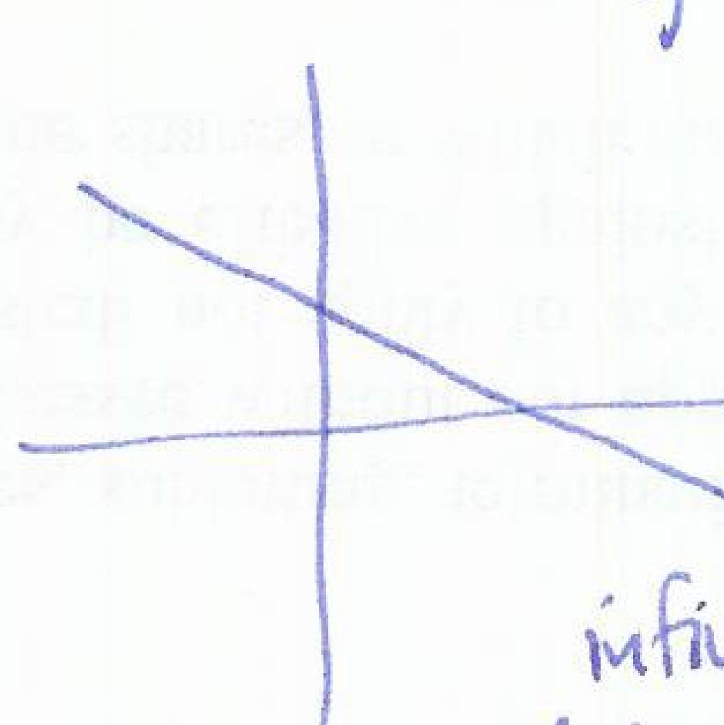
$ax + by = c$
$dx + ey = f$.



unique solution



no solutions (parallel distinct lines)



infinitely many solutions (same line twice)

• want to find <u>all</u> solutions!

② Matrix notation

$$\begin{aligned} 1x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned} \qquad \begin{bmatrix}1 & 2\\ 4 & 5\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}3\\ 6\end{bmatrix}.$$

$$A\underline{x} = \underline{b}$$

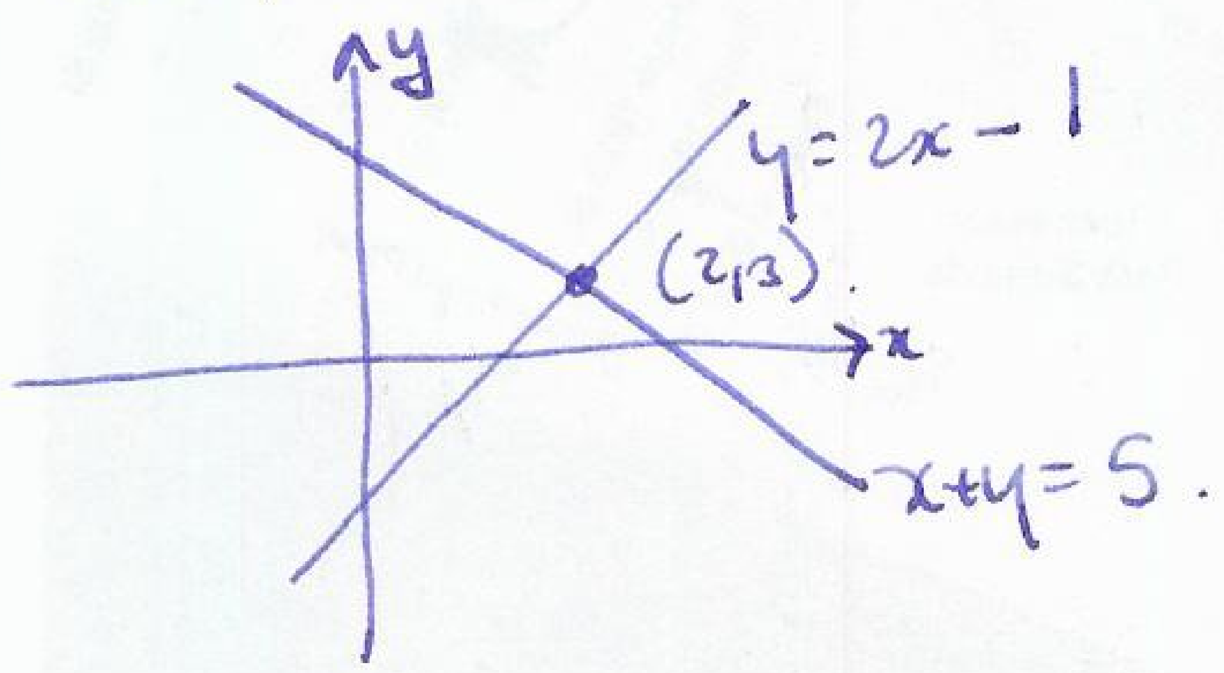
③ - efficiency
- accuracy

§1.2 Geometry of linear equations

Example $2x - y = 1$ 2 equations in 2 unknowns

$x + y = 5$

① row picture: each row describes an equation in $\mathbb{R}^2$, solution set is a line.

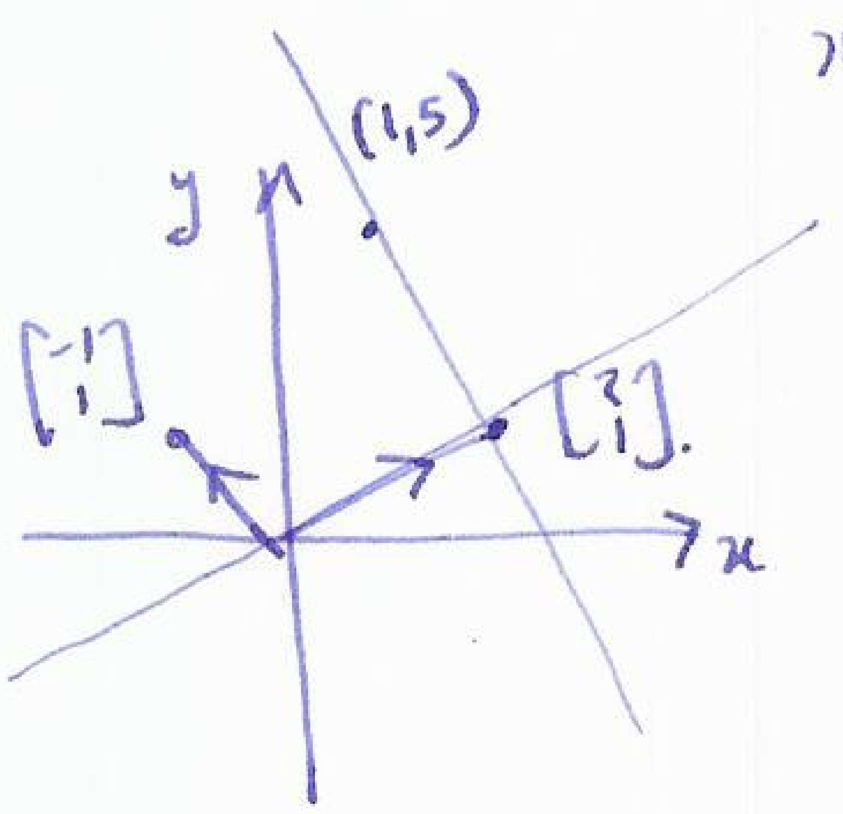


problem: find where two lines intersect.

② column picture: write this as a column vector equation

$$\begin{bmatrix} 2x \\ x \end{bmatrix} + \begin{bmatrix} -y \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

$$x \begin{bmatrix} 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$



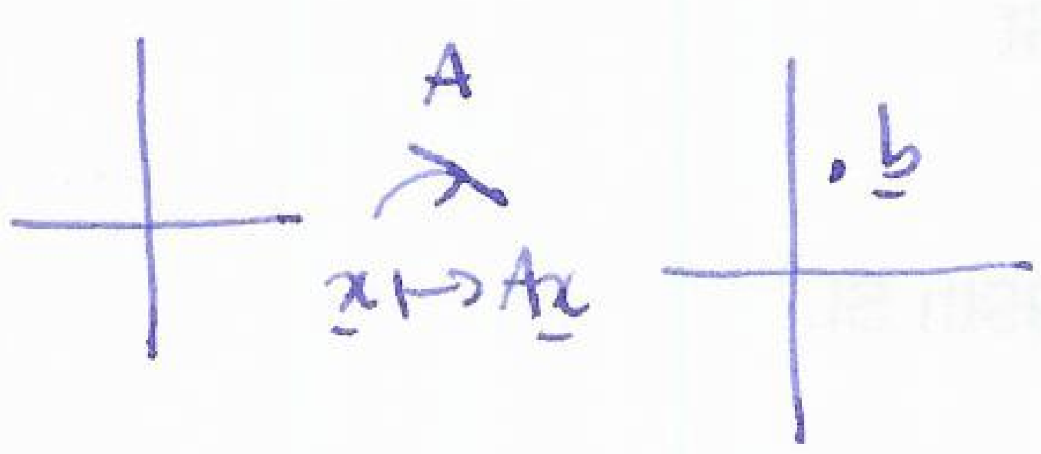
problem: find the (linear) combination of the column vectors on the ~~right~~ left hand side which gives the vector on the right hand side. (i.e. find side length of parallelogram with sides parallel to given vectors)

③ linear map picture

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \underline{x} = \underline{b}$$

A

$A : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ ~~so~~ image of $\mathbb{R}^2$ under A is

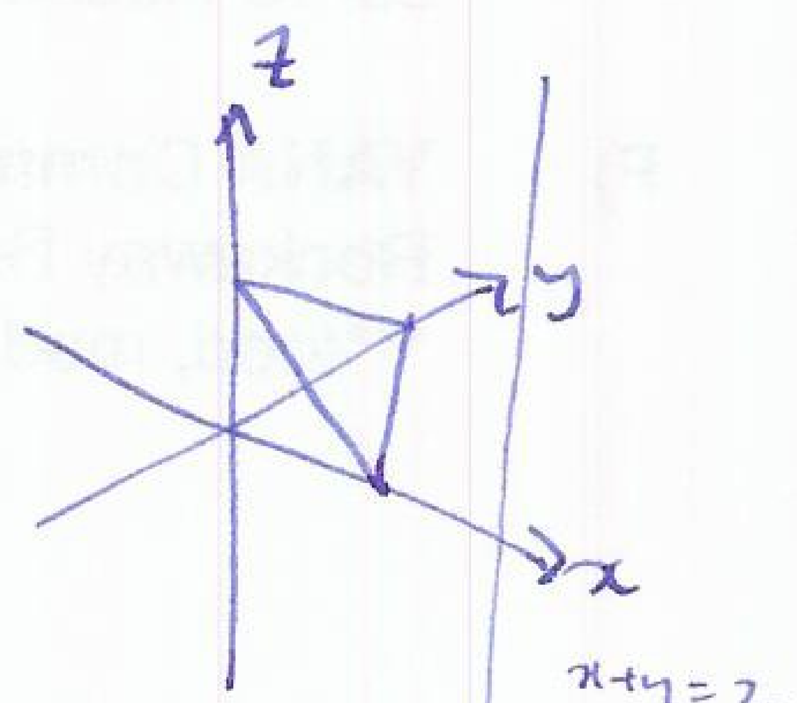


- all of $\mathbb{R}^2$
- a line
- a point

} & does this image contain $\underline{b}$?

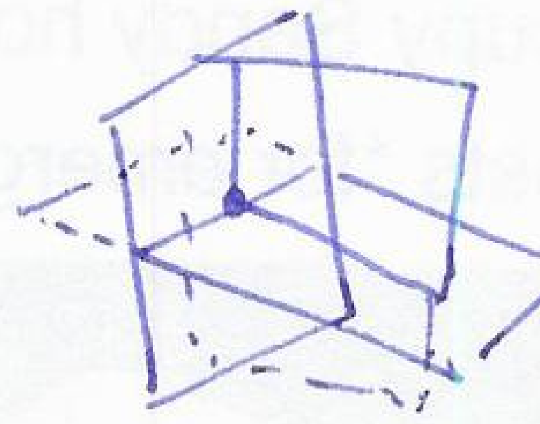
Example 3 equations in 3 unknowns

$$\begin{aligned} x + y + z &= 1 \\ x + y &= 2 \\ x &= 0 \end{aligned}$$



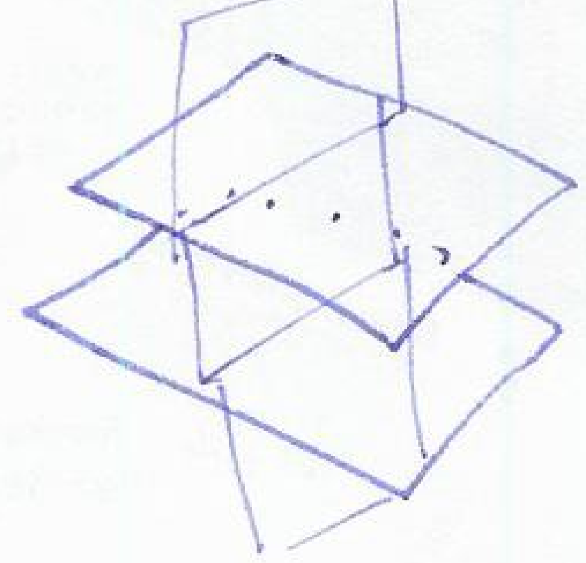
row picture: how can three planes intersect in $\mathbb{R}^3$?

generic picture



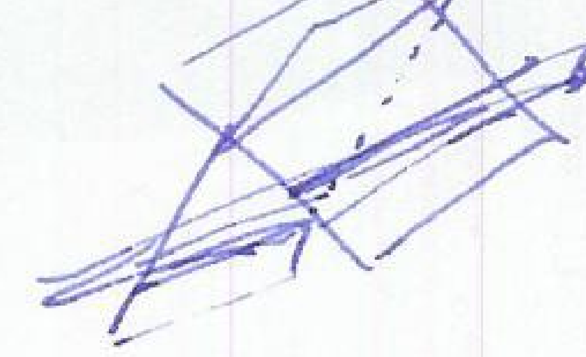
3 planes can intersect in a unique point.

Q: what can go wrong?

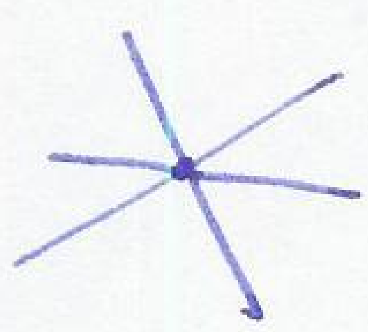


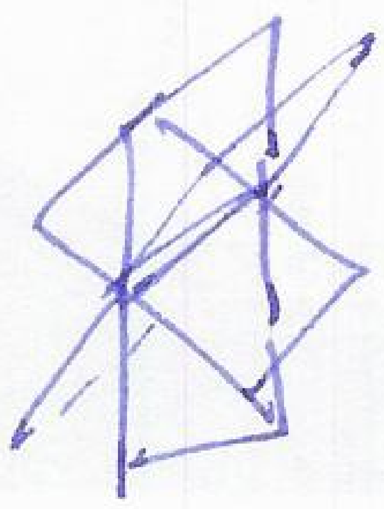
· two parallel planes — no solutions!

· each pair of planes intersects in a parallel distinct line — no solutions!



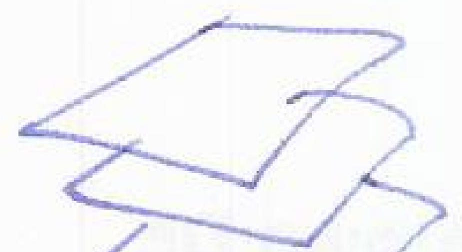
· planes intersect in a common line





infinitely many solutions!

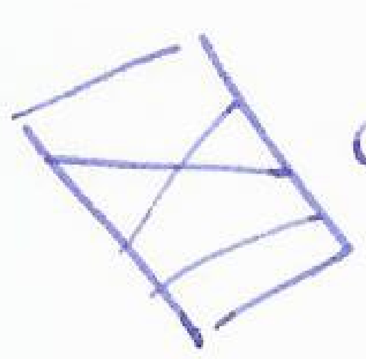
· all planes parallel: no solutions



higher dimensional intuition: 4 equations in 4 unknowns   4 3-planes in $\mathbb{R}^4$

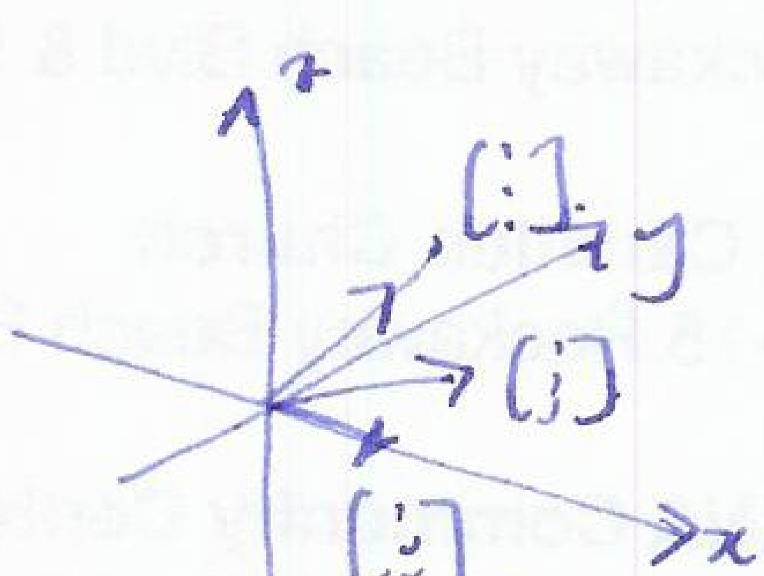
n equations in n unknowns.   n $(n-1)$-planes in $\mathbb{R}^n$

pick one plane $\mathbb{R}^{n-1} \subset \mathbb{R}^n$ and look at intersections.



← copies of $\mathbb{R}^{n-2}$ in $\mathbb{R}^{n-1}$!

column picture $x\begin{bmatrix}1\\1\\1\end{bmatrix} + y\begin{bmatrix}1\\1\\0\end{bmatrix} + z\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\2\\0\end{bmatrix}$



operations on vectors:

– multiplication by scalars $-3\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}-3\\-6\\-9\end{bmatrix}$.

– addition $\begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}3\\4\\5\end{bmatrix} = \begin{bmatrix}4\\6\\8\end{bmatrix}$

a linear combination of the vectors $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\0\end{bmatrix}$ is anything obtained by scalar multiplication and addition. Example: $3\begin{bmatrix}1\\1\\1\end{bmatrix} - 4\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}1\\-1\\3\end{bmatrix}$.