Math 338 Linear Algebra Spring 13 Midterm 3~~a~~ b

Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 3	
Overall	

(1) Let A be the matrix $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$.

(a) Find the eigenvalues of A.

(b) What are the eigenvalues for A^k? Explain your answer.

a) $\det(A - \lambda I) = \begin{vmatrix} -3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = (3+\lambda)\lambda + 2 = \lambda^2 + 3\lambda + 2$

$= (\lambda+2)(\lambda+1) \Rightarrow$ eigenvalues are $-2, -1$

b) $Av = \lambda v \Rightarrow A^k v = A^{k-1} A v = A^{k-1} \lambda v = \lambda A^{k-1} v \Rightarrow A^k v = \lambda^k v$

so A^k has eigenvalues $(-2)^k, (-1)^k$.

(2) Let A be the same matrix as in Q1, i.e. $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$.

(a) Find the eigenvectors for A.

(b) Diagonalize A, i.e. find matrices P and D such that $P^{-1}AP = D$.

a) solve $(A+2I)x = 0$: $\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

solve $(A+I)x = 0$: $\begin{bmatrix} -2 & -1 \\ 2 & 1 \end{bmatrix} \Rightarrow x = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$

b)
$$\begin{tikzcd} \mathbb{R}^2 \arrow[r, "A"] & \mathbb{R}^2 \\ \mathbb{R}^2 \arrow[u, "P"] \arrow[r, "D"'] & \mathbb{R}^2 \arrow[u, "P"'] \end{tikzcd}$$

$P =$ matrix of eigenvectors $= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$

$D = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$

check
$$-\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = -\begin{bmatrix} 4 & 2 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -1 \end{bmatrix}$$

(3) Let A be the same matrix as in Q1, i.e. $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$.

(a) Write down a product of matrices which gives A^k.
(b) Write down a product of matrices which gives e^{At}.
(c) What can you say about e^{At} as $t \to \infty$?

a) $A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} (-2)^k & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

b) $e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$

c) $\begin{bmatrix} e^{-2t} & 0 \\ 0 & e^{-t} \end{bmatrix} \to 0$ as $t \to \infty \Rightarrow e^{At} \to 0$ as $t \to \infty$.

(4) Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ be a basis for $\mathbb{R}^3$, where

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}.$$

Use the Gram-Schmidt process to find an orthonormal basis.

$$q_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$q_2 = \frac{v_2 - (q_1 \cdot v_2) q_1}{\| \quad - \quad \|} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{3}} \, 3 \, \frac{1}{\sqrt{3}}}^{1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$q_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$q_3' = v_3 - (v_3 \cdot q_1) q_1 - (v_3 \cdot q_2) q_2$$

$$= \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{3}} \, 0 \, \frac{1}{\sqrt{3}}}^{0} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} - \overbrace{\frac{1}{\sqrt{2}} \, 2 \, \frac{1}{\sqrt{2}}}^{1} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$q_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(5) (a) Suppose A is an $n \times n$ matrix and $A^2 = A$. What can you say about $\det(A)$?

(b) Suppose A is an $n \times n$ matrix and $\det(A) = 0$. What can you say about the eigenvalues of A?

a) $\det(A^2) = \det(A)^2 = \det(A) \Rightarrow \det(A)^2 - \det(A) = 0$

$\det(A)(\det(A) - 1) = 0$

$\det(A) = 0, 1$.

b) $\det(A)$ = product of eigenvalues $\Rightarrow$ at least one eigenvalues is zero.

(6) Let B be the basis for $\mathbb{R}^2$ given by

$$B = \left\{ \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \end{bmatrix} \right\}.$$

(a) Find a matrix which converts vectors written in the standard basis to vectors written with respect to the basis B.

(b) Use your answer to (a) to write $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (in the standard basis) as a linear combination of vectors in B.

a) $\mathbb{R}^2_{\text{standard}} \to \mathbb{R}^2_B$

$\xleftarrow{P}$

$\xrightarrow{P^{-1}}$

$P = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$

$P^{-1} = -\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$ ← this matrix converts vectors in standard basis to B.

b) $-\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} +5 \\ -3 \end{bmatrix}$

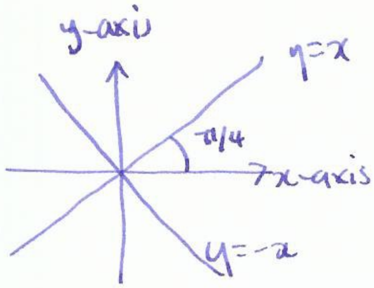
check: $5\begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(7) (a) Write down a matrix A corresponding to an anticlockwise rotation of $\pi/4$ about the origin in $\mathbb{R}^2$.
(b) Write down a matrix B which expands $\mathbb{R}^2$ by a factor of 2 in the x-direction, and a factor of 3 in the y-direction.
(c) Use your answers above to find a matrix which expands $\mathbb{R}^2$ by a factor of 2 in the line $y = x$, and a factor of 3 in the line $y = -x$.

a) rotation by θ: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ $\theta = \pi/4$ $\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = R.$

$$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

b) $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} = E$

c) 

want $R E R^{-1}$

$\{ \frac{1}{\sqrt{2}} \sim -\frac{1}{\sqrt{2}}$

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2 & -3 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 5/2 & -1/2 \\ -1/2 & 5/2 \end{bmatrix}.$$

(8) Let $A = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$.

(a) Find the eigenvalues and eigenvectors for A.

(b) Can you diagonalize A? Explain.

a) $\det(A-\lambda I) = \begin{vmatrix} -\lambda & -1 \\ 1 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 1 = (\lambda-1)^2 \quad \lambda = 1, 1$

~~b)~~ solve $(A-I)x = 0$: $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \Rightarrow$ one eigenvector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

b) no. only one eigenvector

(9) Let $J = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix}$, where x and y may be either 0 or 1.

(a) What are the eigenvalues of A?

(b) What are the largest and smallest number of eigenvectors that A may have?

(c) Suppose $A = PJP^{-1}$, for some invertible matrix P. Show that $A^3 = 0$.

a) 0, 0, 0

b) $1 \leq \#\text{eigenvectors} \leq 3$

c) $J^2 = \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$J^3 = \begin{bmatrix} 0 & 0 & xy \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x & 0 \\ 0 & 0 & y \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$

so $A^3 = PJ^3P^{-1} = P\,0\,P^{-1} = 0$

(10) Let A be a matrix with eigenvalues $\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 2$ and $\lambda_4 = -2$, and the following orthonormal eigenvectors

$$v_1 = \frac{1}{2}\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, v_2 = \frac{1}{2}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix}, v_3 = \frac{1}{2}\begin{bmatrix}-1\\-1\\1\\1\end{bmatrix}, v_4 = \frac{1}{2}\begin{bmatrix}1\\-1\\-1\\1\end{bmatrix}.$$

(a) Write the vector $b = \begin{bmatrix}1\\2\\3\\4\end{bmatrix}$ with respect to the basis of eigenvectors. (Hint: use the fact that the v_i are orthogonal.)

(b) Use your answer above to find Ab with respect to the basis of eigenvectors.

a) $b = c_1 v_1 + c_2 v_2 + c_3 v_3 + c_4 v_4$

$b \cdot v_1 = c_1 \underbrace{v_1 \cdot v_1}_{1} + 0 + 0 + 0.$ $\quad c_1 = b \cdot v_1$ etc.

so $b = \begin{bmatrix} b\cdot v_1 \\ b\cdot v_2 \\ b \cdot v_3 \\ b \cdot v_4 \end{bmatrix} = \frac{1}{2}\begin{bmatrix}5\\-1\\2\\0\end{bmatrix}$

b) $Ab = A(c_1 v_1 + \cdots + c_4 v_4) = \lambda_1 c_1 v_1 + \cdots + \lambda_4 c_4 v_4.$

$= \frac{1}{2}\begin{bmatrix}5\\1\\4\\0\end{bmatrix}$