Math 338 Linear Algebra Spring 13 Midterm 1a

Name: Solutions

- You must do Q1. Do any 7 of the following 9 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	20	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	90	

Midterm 1	
Overall	

(1) Find all solutions to the following set of linear equations.

$$\begin{aligned} x_1 + \ \ x_2 - 2x_3 + 3x_4 &= 6 \\ x_1 + 3x_2 + 2x_3 + 3x_4 &= 8 \\ 2x_1 \qquad\quad - 8x_3 + 5x_4 &= 12 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 6 \\ 1 & 3 & 2 & 3 & 8 \\ 2 & 0 & -8 & 5 & 12 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 6 \\ 0 & 2 & 4 & 0 & 2 \\ 0 & -2 & -4 & -1 & 0 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} \boxed{1} & 1 & -2 & 3 & 6 \\ 0 & \boxed{2} & 4 & 0 & 2 \\ 0 & 0 & 0 & \boxed{-1} & 2 \end{array}\right]$$

$-x_4 = 2 \quad \Rightarrow \quad x_4 = -2$

$x_3 = t$

$2x_2 + 4t = 2 \quad \Rightarrow \quad x_2 = 1 - 2t$

$x_1 + (1-2t) - 2t - 6 = 6 \quad \Rightarrow \quad x_1 = 11 + 4t$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11 \\ 1 \\ 0 \\ -2 \end{bmatrix} + t \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}$$

(2) Suppose A and B are invertible matrices.

(a) Is ABA^{-1} invertible? If so, write down the inverse.

(b) Is $A+B$ invertible? Explain why or give a counterexample.

a) $(ABA^{-1})^{-1} = AB^{-1}A^{-1}$

b) no $A = I$, $B = -I$ $A+B = I - I = 0$ not invertible

(3) Give an example of a system of three equations in two unknowns which is inconsistent.

$x + y = 0$

$x + y = 1$

$x + y = 2$

(4) (a) If a system has three equations and four unknowns can it be inconsistent? Give an example or justify your answer.

(b) If a system has three equations and four unknowns can there be a unique solution? Justify your answer.

a) yes

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + x_2 + x_3 + x_4 = 1$$
$$x_1 + x_2 + x_3 + x_4 = 2$$

b) no. There may be at most 3 pivots, so there is a free variable, so if there is a solution, there are infinitely many.

(5) Consider the matrix

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

(a) Describe in words the row operations corresponding to left multiplication by L.
(b) Write down L^{-1}.

a) ① keep row ① the same

② − ① subtract row ① from row ②

③ − 3① subtract 3① from row ③

b) $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}$

(6) Use row operations to find the row echelon form the following matrix, writing out clearly what row operations you used.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -3 & 0 \\ 1 & 3 & 1 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② + ① \\ ③ - ① \end{array} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{array}{c} ① \\ ② \\ ③ + ② \end{array} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

(7) Find the LU factorization for the matrix A in the previous question. Hint: use your answer to the previous question.

$$L_2 L_1 A = U \Rightarrow A = \underbrace{L_1^{-1} L_2^{-1}}_{L} U$$

where $L_1 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ $\quad L_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$.

$L_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ $\quad L_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 1 & -1 & 1 \end{bmatrix} = L$$

(9) Find a 3×3 matrix A such that $A \neq I$ but $A^2 = I$.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(8) Are the following vectors linearly independent?

$$\{[1, 2, 1], [-1, -3, 0], [1, 3, 1]\}$$

Hint: use your solution to Q6 or Q7.

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 0 \\ 1 & 3 & 1 \end{bmatrix} \underset{\text{ref}}{\leadsto} \begin{bmatrix} 1 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

3 pivots $\Rightarrow$ linearly independent

(10) Use row operations to find the inverse of the following matrix.

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & -1 & 4 \end{bmatrix}$$

Check your answer.

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 1 & -2 & 4 & 0 & 1 & 0 \\ 1 & -1 & 4 & 0 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 2 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right]$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & +1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right]$$

$$\cancel{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & -1/3 & -2/3 \\ 0 & 1 & 0 & 0 & -1 & +1 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array}\right]}$$

check

$$\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ 0 & -1 & +1 \\ -1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

check

$$\cancel{\begin{bmatrix} 1 & -1 & 2 \\ 1 & -2 & 4 \\ 1 & -1 & 4 \end{bmatrix} \begin{bmatrix} 2 & -1/3 & -2/3 \\ 0 & -1 & +1 \\ -1/2 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$