Practice Questions

Q1 a) $T(u,v) = (u,v,0) \quad u^2+v^2 \le 1$

or $T(r,\theta) = (r\cos\theta, r\sin\theta, 0) \quad 0\le r \le 1, \; 0\le\theta\le 2\pi$

b) $c(\theta) = (\cos\theta, \sin\theta, 0) \quad 0\le\theta\le 2\pi$

c) $T(u,v) = (u,v,4-u-2v) \quad u^2+v^2\le 1$

or $T(r,\theta) = (r\cos\theta, r\sin\theta, 4-r\cos\theta-2r\sin\theta) \quad 0\le r\le 1 \quad 0\le\theta\le 2\pi.$

d) $c(\theta) = (\cos\theta, \sin\theta, 4-\cos\theta-2\sin\theta)$

e) $T(\theta,z) = (\cos\theta, \sin\theta, z) \qquad 0\le z\le 4-\cos\theta-2\sin\theta.$
$\quad 0\le\theta\le 2\pi$

Q2 $\underline{F} = \langle z,x,y\rangle$

a) $\iint_S \text{curl}(\underline{F})\, d\underline{S} = \int_{\partial S} \underline{F}\cdot d\underline{s}$

RHS: ∂S is b) above: $\int_0^{2\pi} \langle 0, \cos\theta, \sin\theta\rangle\cdot\langle -\sin\theta, \cos\theta, 0\rangle \, d\theta$

$c'(\theta) = (-\sin\theta, \cos\theta, 0)$ $\qquad = \int_0^{2\pi} \cos^2\theta \, d\theta = \pi$

LHS: S = top ∪ vertical = c) ∪ e) $\qquad \text{curl}(\underline{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \langle 1,1,1\rangle$

c): $\int_0^{2\pi}\int_0^1 \langle 1,1,1\rangle\cdot n(r,\theta)\, dr\, d\theta$

$\frac{\partial T}{\partial r} = (\cos\theta, \sin\theta, -\cos\theta-2\sin\theta)$

$\frac{\partial T}{\partial \theta} = (-r\sin\theta, r\cos\theta, r\sin\theta-2r\cos\theta)$

$\underline{n} = \langle r\sin^2\theta - 2r\sin\theta\cos\theta + r\cos^2\theta + 2r\sin\theta\cos\theta, \; -(r\sin\theta\cos\theta - 2r\cos^2\theta + r\sin\theta\cos\theta + 2r\sin^2\theta), \; r\cos^2\theta + r\sin^2\theta\rangle.$

$\int_0^{2\pi}\int_0^1 r+2r+r \, dr\, d\theta = 4\pi$

e) $\frac{\partial T}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$

$\frac{\partial T}{\partial z} = (0,0,1)$

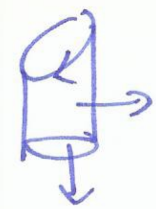
$\underline{n} = \langle \cos\theta, \sin\theta, 0\rangle$

$$\int_0^{2\pi}\int_0^{4-\cos\theta-2\sin\theta} \cos\theta+\sin\theta \, dz\, d\theta = \int_0^{2\pi}(\cos\theta+\sin\theta)(4-\cos\theta-2\sin\theta)\, d\theta$$

$$= \int_0^{2\pi} 4\cos\theta - \cos^2\theta - 2\sin\theta\cos\theta + 4\sin\theta - \cos\theta\sin\theta - 2\sin^2\theta \, d\theta$$

$$= \int_0^{2\pi} -1 - \sin^2\theta \, d\theta = -3\pi$$

$\underset{LHS}{4\pi - 3\pi} = \underset{RHS}{\pi}$, as required.



b) $\iint_S \mathrm{curl}(\underline{F})\, d\underline{S} = \int_{\partial S} \underline{F}.d\underline{s}$

RHS: ∂S in d) above: $c'(\theta) = (-\sin\theta, \cos\theta, \sin\theta - 2\cos\theta)$

$$\int_0^{2\pi} \langle 0, \cos\theta, \sin\theta\rangle . \langle -\sin\theta, \cos\theta, \sin\theta - 2\cos\theta\rangle \, d\theta$$

$= \int_0^{2\pi} \cos^2\theta + \sin^2\theta - 2\sin\theta\cos\theta \, d\theta = 2\pi$; answer is -2π (wrong orientation)

LHS: $S = \underset{a)}{\text{bottom}} \cup \underset{e)}{\text{vertical}}$ e) is -3π from above.

a):

$$\left.\begin{aligned}\frac{\partial T}{\partial r} &= (\cos\theta, \sin\theta, 0)\\ \frac{\partial T}{\partial \theta} &= (-\sin\theta, \cos\theta, 0)\end{aligned}\right\}\ \underline{n} = \langle 0, 0, r\rangle$$

$$\int_0^{2\pi}\int_0^1 r\, dr\, d\theta = \pi$$

$\underset{LHS}{-2\pi} = \underset{RHS}{-3\pi + \pi}$ as required.

c) $\mathrm{div}(\underline{F}) = 0$ so $\iiint_W \mathrm{div}(\underline{F})\, dV = 0$.

$S = \underset{a)}{\text{bottom}} \cup \underset{c)}{\text{top}} \cup \underset{e)}{\text{vertical}}$.

a): $\int_0^{2\pi}\int_0^1 \langle 0, r\cos\theta, \sin\theta\rangle . \langle 0,0,r\rangle \, dr\, d\theta = \int_0^{2\pi}\int_0^1 r^2 \sin\theta \, dr\, d\theta = 0$

c):

$$\int_0^{2\pi}\int_0^1 \langle 4 - r\cos\theta - r\sin\theta,\ r\cos\theta,\ r\sin\theta\rangle \cdot \langle r, 2r, r\rangle \, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 4r - r^2\cos\theta - r^2\sin\theta + 2r^2\cos\theta + r^2\sin\theta \, dr\, d\theta = \int_0^{2\pi}\int_0^1 4r \, dr\, d\theta = 4\pi.$$

e):

$$\int_0^{2\pi}\int_0^{4-\cos\theta-2\sin\theta} \langle z, \cos\theta, \sin\theta\rangle \cdot \langle \cos\theta, \sin\theta, 0\rangle \, dz\, d\theta$$

$$= \int_0^{2\pi}\int_0^{4-\cos\theta-2\sin\theta} z\cos\theta + \sin\theta\cos\theta \, dz\, d\theta$$

$$= \int_0^{2\pi} \tfrac{1}{2}(4-\cos\theta-2\sin\theta)^2\cos\theta + (4-\cos\theta-2\sin\theta)\sin\theta\cos\theta \, d\theta.$$

$$= \int_0^{2\pi} \tfrac{1}{2}\Big(\overset{0}{16\cos\theta} + \overset{0}{\cos^3\theta} + \overset{0}{4\sin^2\theta\cos\theta} - 8\cos^2\theta - \overset{0}{16\sin\theta\cos\theta} + \overset{0}{4\sin\theta\cos^2\theta}\Big)$$
$$+ \underset{0}{4\sin\theta\cos\theta} - \underset{0}{\sin\theta\cos^2\theta} - \underset{0}{2\sin^2\theta\cos\theta} \, d\theta \quad = -4\pi.$$

so $0 = 4\pi - 4\pi$ as required.