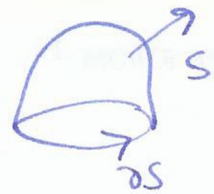
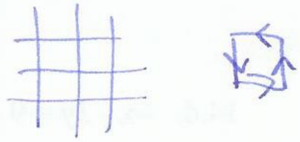
intuition

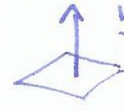


$$\oint_{\partial S} \underline{F}\cdot d\underline{s} = \iint_S \mathrm{curl}(\underline{F})\cdot d\underline{S}$$

chop into small bits:



same calculation as last time



$\underline{n} = (0,0,1)$.

$$\nabla\times\underline{F}\cdot(\underline{n}) \approx \left(0,0,\frac{\partial F_2}{\partial x}-\frac{\partial F_1}{\partial y}\right)\cdot \square$$

Example $\underline{F} = \langle -y, 2x, x+z\rangle$

$S =$ upper hemisphere

$\oint_{\partial S} \underline{F}\,d\underline{s}$

$\underline{c}(\theta) = (\cos\theta, \sin\theta, 0)$.

$\underline{c}'(\theta) = (-\sin\theta, \cos\theta, 0)$.

$$\oint_{\partial S} \underline{F}\cdot d\underline{s} = \int_0^{2\pi} (-\sin\theta, 2\cos\theta, \cos\theta)\cdot(-\sin\theta,\cos\theta,0)\cdot d\theta.$$

$$= \int_0^{2\pi} \sin^2\theta + 2\cos^2\theta\, d\theta = \cancel{2\pi}\int_0^{2\pi} 1+\cos^2\theta\, d\theta = 2\pi+\pi = 3\pi.$$

$\iint_S \mathrm{curl}(\underline{F})\cdot d\underline{S}$

$$\nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & 2x & x+z \end{vmatrix} = \langle 0, -1, 2+1\rangle = \langle 0,-1,3\rangle$$

parameterization: $(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$ $\quad 0\le\theta\le 2\pi$, $0\le\phi\le\pi/2$

$$\int_0^{\pi/2}\int_0^{2\pi} (0,-1,3)\cdot\underline{n}\; d\theta\, d\phi$$

$$\frac{\partial\Phi}{\partial\theta} = (-\sin\theta\sin\phi, \cos\theta\sin\phi, 0)$$

$$\frac{\partial\Phi}{\partial\phi} = (\cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi)$$

$$\underline{n} = \langle \cos\theta\sin^2\phi, + \sin\theta\sin^2\phi, \sin^2\theta\sin\phi\cos\phi + \cos^2\theta\cos\phi\sin\phi\rangle.$$

$$= \sin\phi\langle\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi\rangle$$

$$\int_0^{\pi/2}\int_0^{2\pi} -\sin\theta\sin^2\phi + 3\cos\phi\sin\phi\, d\theta\, d\phi = 3\pi.$$

recall: $\underline{F} = \nabla\phi$ then $\int_{C_1} \underline{F}.\underline{ds} = \int_{C_2} \underline{F}.\underline{ds} = \phi(Q) - \phi(P)$

for any paths C_1, C_2 from P to Q (path independence for gradient vector fields, fundamental theorem for gradient vector fields).

Stoke's thm: $\oint_{\partial S} \underline{F}.\underline{ds} = \iint_S \text{curl}(\underline{F}).\underline{dS}$ note integral only depends on ∂S, not S itself

i.e. surface independence for curl vector fields.

S_2 S_1 C_1.

$$\oint_{\partial S} \underline{F}.\underline{ds} = \iint_{S_1} \text{curl}(\underline{F}).\underline{dS} = \iint_{S_2} \text{curl}(\underline{F}).\underline{dS} \quad \text{if } \partial S_1 = \partial S_2.$$

in particular if S is closed ($\partial S = \phi$) then $\iint_S \text{curl}(\underline{F}).\underline{dS} = 0$.

S  $S = S_1 \cup S_2$ with $\partial S_1 = -\partial S_2$ (reverse orientation).

§17.3 Divergence Theorem

recall we have theorems of the form: integral of a derivative on a domain = integral over oriented boundary of domain.

FTC: $\int_a^b f'(x)\,dx = f(b) - f(a)$ a b $[a,b]$. $\partial[a,b] = +b - a$.

Line integrals: $\int_C \underline{\nabla\phi}.\underline{ds} = \phi(Q) - \phi(P)$  P C Q

Stokes' Thm: $\iint_S \text{curl}(F).\underline{dS} = \int_{\partial S} \underline{F}.\underline{ds}$  S ∂S.

Divergence thm: $\iiint_W \text{div}(\underline{F}).dV = \iint_{\partial W} \underline{F}.\underline{dS}$  W outward pointing normal

Defn the divergence of a vector field $\underline{F}$, $\text{div}(\underline{F}) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$ scalar!

$\text{div}(\underline{F}) = \underline{\nabla}\cdot\underline{F}$

useful properties $\nabla\cdot(\underline{F}+\underline{G}) = \nabla\cdot\underline{F} + \nabla\cdot\underline{G}$

$\nabla\cdot(c\underline{F}) = c\,\nabla\cdot\underline{F}$ (c constant)

Example $\underline{F} = \langle x^2+y^2, xyz, e^{xyz}\rangle$

$\nabla\cdot\underline{F} = 2x + xz + xye^{xyz}$.

Intuition: small ball B, $\partial B = S$, so $\operatorname{div}(\underline{F})$ approx constant.

then flux across $S = \iiint_B \operatorname{div}(\underline{F})\,dV \approx \operatorname{div}(F)\operatorname{vol}(B)$.

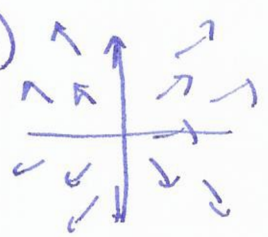
flux $\approx$ divergence $\times$ vol. (locally).

- $\operatorname{div}(\underline{F}) > 0$ at point : fluid flows out "source"

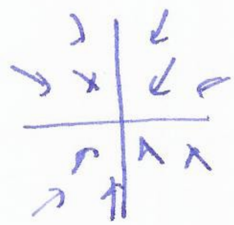
- $\operatorname{div}(\underline{F}) < 0$: fluid flows in "sink"

- $\operatorname{div}(\underline{F}) = 0$: no net flow in or out "incompressible flow".

Examples (2d)

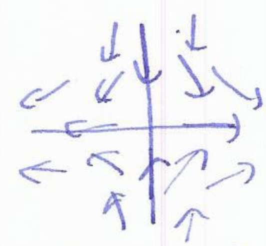


$\underline{F} = \langle x,y\rangle$ $\nabla\cdot\underline{F} = 2$



$\underline{F} = \langle -x,-y\rangle$

$\nabla\cdot\underline{F} = -2$



$\underline{F} = \langle x,-y\rangle$

$\nabla\cdot\underline{F} = 0$.

Example (verify Stokes')

$\underline{F} = \langle y, yz, z^2\rangle$ S: $x^2+y^2 = 4$, $0 \le z \le 5$

$\nabla\cdot\underline{F} = 0 + z + 2z = 3z$.

$\iiint_W 3z\,dV$ use cylindricals $= 150\pi$.