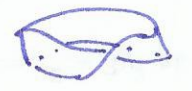
Warning: can only integrate vector functions over orientable surfaces.

orientable  non-orientable:  Möbius band.

Example find $\iint_S \underline{F}\cdot d\underline{S}$ $\quad \underline{F} = \langle 0,0,x\rangle$

$$S : \Phi(u,v) = (u^2, v, u^3 - v^2) \qquad \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le 1 \end{matrix}$$

$$\frac{\partial \Phi}{\partial u} = \langle 2u, 0, 3u^2 \rangle \qquad n = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{vmatrix} = \langle -3u^2, 4uv, 2u \rangle$$

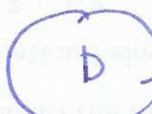
$$\frac{\partial \Phi}{\partial v} = \langle 0, 1, -2v \rangle$$

$$\iint_S \underline{F}\cdot d\underline{S} = \int_0^1 \int_0^1 \underline{F}(\Phi(u,v))\cdot \underline{n}(u,v)\, dv\, du$$

$$= \int_0^1\int_0^1 \overset{u^2}{\cancel{(u^3 - v^2)}}\, 2u \, dv\, du = \int_0^1 2u^3\, du = \left[\frac{1}{2}u^4\right]_0^1 = \frac{1}{2}.$$

§17.1 Green's Theorem

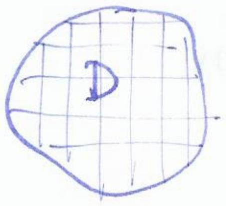
recall: $\oint_C \underline{\nabla} f \cdot ds = 0.$  C $\quad$ suppose $\underline{F} \ne \underline{\nabla} f$?

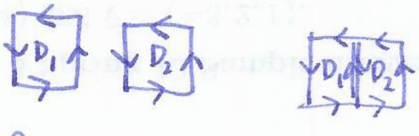
orientation convention:  D C counter-clockwise.

Thm (Green's Thm) D domain s.t. $\partial D = C$ simple closed curve oriented clockwise

then $\oint_C \underline{F}\cdot d\underline{s} = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA \qquad \underline{F} = \langle F_1, F_2 \rangle. \qquad \square.$

~~Intuition~~: intuition

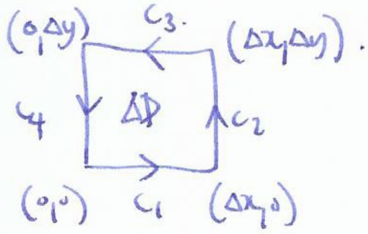


note: 

$$\oint_{\partial D_1} + \oint_{\partial D_2} = \oint_{\partial (D_1 \cup D_2)}$$

so suffices to check on a very small square where $\underline{F}$ is almost linear.

$$\underline{F} = \langle px+qy, rx+sy \rangle$$



$$c_1: \int_0^{\Delta x} pt + q.0 \, dt = \left[\tfrac{1}{2}pt^2\right]_0^{\Delta x} = \tfrac{1}{2}p\Delta x^2$$

$$c_2: \int_0^{\Delta y} r\Delta x + st \, dt = \left[r\Delta x t + \tfrac{1}{2}st^2\right] = r\Delta x\Delta y + \tfrac{1}{2}s\Delta y^2$$

$$c_3: -\int_0^{\Delta x} pt + q\Delta y \, dt = -\left[\tfrac{1}{2}pt^2 + q\Delta y t\right]_0^{\Delta x} = -\tfrac{1}{2}p\Delta x^2 - q\Delta x\Delta y$$

$$c_4: -\int_0^{\Delta y} r.0 + st \, dt = -\left[\tfrac{1}{2}st^2\right]_0^{\Delta y} = -\tfrac{1}{2}s\Delta y^2.$$

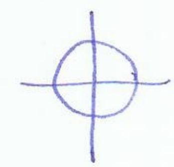
$$\oint_{\Delta D} \underline{F}.\underline{ds} = r\Delta x\Delta y - q\Delta x\Delta y = (r-q)\Delta A = \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\Delta A$$

$$\oint_D \underline{F}.ds = \sum (r-q)\Delta A = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dA. \qquad \square$$

<u>Example</u> $\oint_C \langle xy^2, x \rangle . \underline{ds}$ $\quad$ $C=$ unit circle oriented counterclockwise.

<u>Line integral</u> $c(\theta) = (\cos\theta, \sin\theta) \quad 0 \leq \theta \leq 2\pi$

$c'(\theta) = (-\sin\theta, \cos\theta)$



$$\int_0^{2\pi} (\cos\theta\sin^2\theta, \cos\theta)(-\sin\theta, \cos\theta)\, d\theta = \int_0^{2\pi} -\cos\theta\sin^3\theta + \cos^2\theta \, d\theta$$

$$= \left[\frac{-\sin^4\theta}{4} + \tfrac{1}{2}\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} = \pi$$

<u>double integral</u> $\iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} dA = \iint_D 1 - \underbrace{2xy}_{\iint_D 2xy\, dA = 0 \text{ by symmetry}} dA = \iint_D 1\, dA = \text{area} = \pi. \quad \square$

Multiple boundary components : orient components

think:

§17.2 Stoke's Theorem (3D version of Green's theorem).

Thm $\displaystyle\oint_{\partial S} \underline{F}\cdot d\underline{s} = \iint_S \underbrace{\nabla\times\underline{F}}_{\text{curl}(\underline{F})}\cdot d\underline{S}$.

S surface (oriented).

∂S boundary (induced orientation).

remark if S is closed ($\partial S=\phi$) then $\oint_\phi \underline{F}\cdot d\underline{s}=0$

$$\text{curl}(\underline{F}) = \nabla\times\underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, -\frac{\partial F_3}{\partial x} + \frac{\partial F_1}{\partial z}, \underbrace{\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}}_{\text{Green's thm!}} \right\rangle.$$

vector!

Example $\underline{F} = \underline{x} = \langle x,y,z\rangle$ $\nabla\times\underline{F} = \underline{0}$

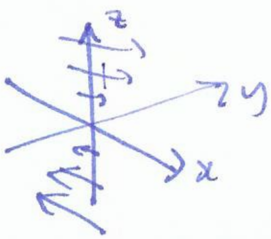
$$\underline{F} = \langle xy, e^x, \tfrac{y}{z}\rangle \quad \nabla\times\underline{F} = \begin{vmatrix} i & j & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & e^x & \frac{y}{z} \end{vmatrix} = \langle \tfrac{1}{z}-0, -0+0, e^x - x\rangle$$

intuition what does curl($\underline{F}$) measure?

fluid flow: imagine flowing along the vector field:

curl($\underline{F}$) = $(\frac{1}{2})$ angular velocity of small paddle wheel.

Example $\underline{F} = \langle y,0,0\rangle$. $\nabla\times\underline{F} = \langle 0,0,1\rangle$



rotational force.