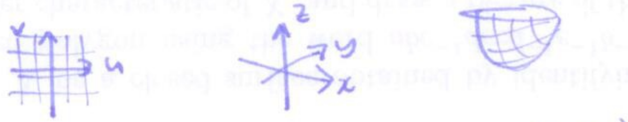
# §16.4 Paramaterized surfaces and surface integrals



recall: parameterized curve $\underline{c}(t) : \mathbb{R} \longrightarrow \mathbb{R}^3$

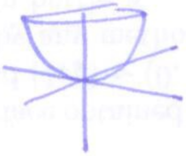
$t \longmapsto (x(t), y(t), z(t))$

parameterized surface $\phi : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$




$(u,v) \longmapsto (x(u,v), y(u,v), z(u,v))$

Examples ① paraboloid $z = x^2 + y^2$



$\phi(u,v) = (u, v, u^2 + v^2)$.

note we can parameterize any graph $z = f(x,y)$ by $(u,v) \mapsto (u, v, f(u,v))$

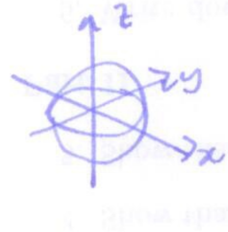
② cylinder $x^2 + y^2 = 1$


$(\theta, z) \longmapsto (\cos\theta, \sin\theta, z)$

$0 \leq \theta \leq 2\pi$

③ sphere

$x^2 + y^2 + z^2 = 1$



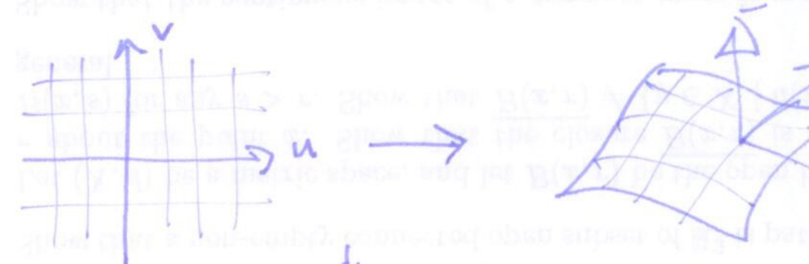

spherical coords!

$\rho = 1 \quad (\theta, \phi) \longrightarrow (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$

$0 \leq \theta \leq 2\pi$

$0 \leq \phi \leq \pi$

coordinate lines




$\frac{\partial \phi}{\partial v}$

$\frac{\partial \phi}{\partial u}$

$(u,v) \xrightarrow{\phi} (x(u,v), y(u,v), z(u,v))$

$\frac{\partial \phi}{\partial u} = \left(\frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, \frac{\partial z}{\partial u}\right)$ = tangent vector to surface in u-direction

$\frac{\partial \phi}{\partial v} = \left(\frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, \frac{\partial z}{\partial v}\right)$ = tangent vector to surface in v-direction

Q: how do we find the normal vector to the surface?

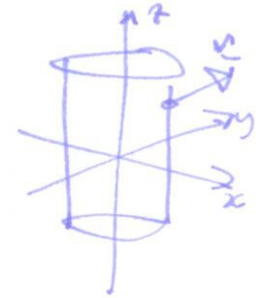
A: $\underline{n} = \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v}$ is a normal vector.

**Example** cylinder $(\theta, z) \xrightarrow{\phi} (\cos\theta, \sin\theta, z)$

$$\frac{\partial \phi}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$$

$$\frac{\partial \phi}{\partial z} = (0, 0, 1)$$

$$\underline{n} = \frac{\partial \phi}{\partial \theta} \times \frac{\partial \phi}{\partial z} = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle \cos\theta, \sin\theta, 0 \rangle$$



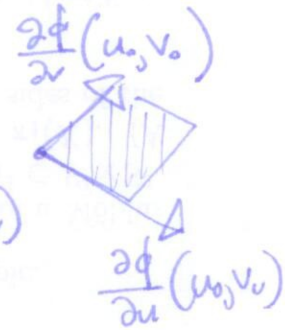

**Surface area**

$\phi: \mathbb{R}^2 \longrightarrow \mathbb{R}^3.$


need scaling function for this piece.

linear approx

$\Delta A = \Delta u \Delta v$




we can use area of parallelograms

$$= \left\| \frac{\partial \phi}{\partial u} \times \frac{\partial \phi}{\partial v} \right\| = \underline{n}(u, v).$$

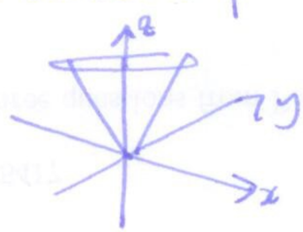
so surface area: $\text{area}(S) = \iint_R \| \underline{n}(u, v) \| \, du\, dv$

How to integrate a scalar function $f(x, y, z)$ over $S$:

$$\iint_S f(x, y, z)\, dS = \iint_R f(x, y, z) \, \| \underline{n}(u, v) \| \, du\, dv$$

$\uparrow$ S with same choice of parameterization $(u, v)$.

**Example** find surface area of cone.




$\phi(\theta, t) = (t\cos\theta, t\sin\theta, t)$

$0 \leq \theta \leq 2\pi \quad 0 \leq t \leq 2$

$$\frac{\partial \phi}{\partial t} = (\cos\theta, \sin\theta, 1)$$

find $\underline{n}$:

$$\frac{\partial \phi}{\partial \theta} = (-t\sin\theta, t\cos\theta, 0)$$

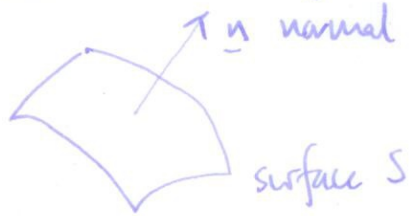
$$\underline{n} = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -t\sin\theta & t\cos\theta & 0 \end{vmatrix} = \langle -t\cos\theta, -t\sin\theta, t \rangle$$

$$\|\underline{n}\| = \sqrt{2t^2} = \sqrt{2}\, t$$

$$\text{area} = \int_0^{2\pi} \int_0^2 1 \cdot \sqrt{2}\, t \, dt \, d\theta \Rightarrow \left[ \frac{\sqrt{2}\, t^2}{2} \right]_0^2 = 2\sqrt{2}.$$

$$\int_0^{2\pi} 2\sqrt{2} \, d\theta = 4\sqrt{2}\,\pi.$$

## §16.5 Vector integrals over surfaces




integrate a vector field $\underline{F}$ over $S$:

notation: $\displaystyle \iint_S \underline{F} \cdot d\underline{S} = \iint_S (\underline{F} \cdot \underline{\hat{n}}) \, dS$

$\underline{\hat{n}}$ = unit normal

this is also called the flux of $\underline{F}$ across $S$.

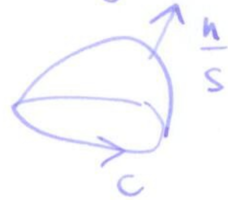
in terms of a parameterization $\phi(u,v)$ for $S$: $\displaystyle \iint_D F(\phi(u,v)) \cdot \underline{n}(u,v) \, du \, dv$

**Example** $\underline{F}$ = fluid flow

$\displaystyle \iint_S \underline{F} \cdot d\underline{S}$ = amount of fluid flowing through surface


**Electricity** $\underline{E}$ electric field

$\underline{B}$ magnetic field




Faraday's law of induction: $\displaystyle \int_C \underline{E} \cdot d\underline{s} = -\frac{d}{dt} \iint_S \underline{B} \cdot d\underline{S}$