$$\underline{c}'(t) = \langle -3\sin\theta,\ 2\cos\theta \rangle$$

$$\int_0^{2\pi} \underline{F}\cdot d\underline{s} = \int_0^{2\pi} \langle 6+4\sin\theta, -3 \rangle \cdot \langle -3\sin\theta, 2\cos\theta \rangle \, d\theta$$

$$= \int_0^{2\pi} -18\sin\theta - 12\sin^2\theta - 6\cos\theta \, d\theta = -12\int_0^{2\pi} \sin^2\theta \, d\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = 1 - 2\sin^2\theta$$

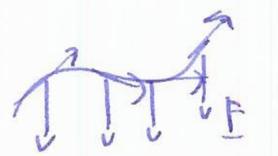
$$= \int_0^{2\pi} -6 + 6\cos 2\theta \, d\theta = -6\int_0^{2\pi} d\theta = -12\pi.$$

<u>useful properties</u> $\displaystyle\int_C (\underline{F}+\underline{G})\cdot d\underline{s} = \int_C \underline{F}\cdot d\underline{s} + \int_C \underline{G}\cdot d\underline{s}$

$$\int_C k\underline{F}\cdot d\underline{s} = k\int_C \underline{F}\cdot d\underline{s}$$

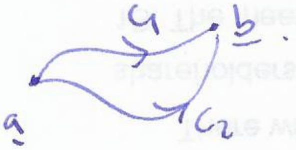
reverse orientation on C: $\displaystyle\int_C \underline{F}\cdot d\underline{s} = -\int_{-C} \underline{F}\cdot d\underline{s}$

<u>Physical interpretation</u> work done $\displaystyle W = \int_C \underline{F}\cdot d\underline{s}$



<u>§16.3 Conservative vector fields</u>

In general $\displaystyle\int_C \underline{F}\cdot d\underline{s}$ depends on the path C, not just the endpoints



$$\int_{C_1} \underline{F}\cdot d\underline{s} \neq \int_{C_2} \underline{F}\cdot d\underline{s}$$

However, for special vector fields $\underline{F}$, $\int_C \underline{F}\cdot d\underline{s}$ only depends on the endpoints. These vector fields are called <u>conservative</u>, i.e. if C_1 and C_2 are two paths with the same endpoints

$$F \text{ conservative} \Rightarrow \int_{C_1} \underline{F}\cdot d\underline{s} = \int_{C_2} \underline{F}\cdot d\underline{s}.$$

special case C is a closed curve then $\underline{F}$ conservative $\Rightarrow \int_C \underline{F}\cdot d\underline{s} = 0$

(notation: sometimes written $\oint_C \underline{F}\cdot d\underline{s}$).

recall if $\underline{F}$ is a gradient vector field then $\underline{F} = \nabla f$ for some scalar function f.

Thm (fundamental theorem for gradient vector fields)
If $\underline{F} = \nabla f$ on a domain D, then for every oriented curve C in D with initial point P and final point Q $\int_C \underline{F}\cdot d\underline{s} = f(Q) - f(P)$

(if C is closed $P = Q$ so $\oint_C \underline{F}\cdot d\underline{s} = 0$).

Thm Every conservative vector field $\underline{F}$ on an (open connected) domain D is a gradient vector field, i.e. $\underline{F} = \underline{\nabla} f$ for some f.

Proof (sketch)

pick a point $\underline{x} \in D$ and define

$$f(\underline{y}) = \int_C \underline{F}\cdot d\underline{s} \quad \text{where } C \text{ is any path from}$$

$\underline{x}$ to $\underline{y}$.. Note: this is well defined if $\underline{F}$ is conservative!

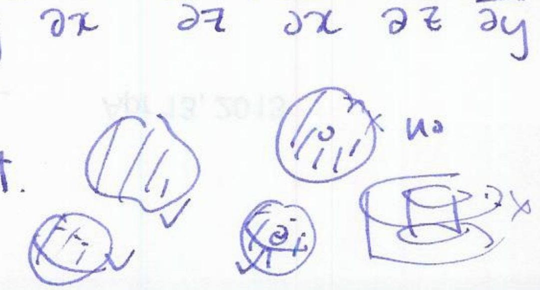
now compute $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$ by choosing short horizontal / vertical paths $\square$.

Q: when is $\underline{F}$ conservative / have a potential function?

Thm Let $\underline{F} = \langle F_1, F_2, F_3 \rangle$ be a vector field on a simply connected domain D. Then if the cross partials are equal $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$ $\frac{\partial F_1}{\partial z} = \frac{\partial F_3}{\partial x}$ $\frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}$

then $\underline{F} = \nabla f$ for some f.

simply connected: every loop can be shrunk to a point.



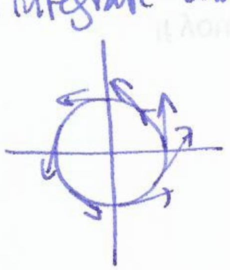
Example Vortex vector field.

$$\underline{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$$

(1) mixed partials equal

$$\left.\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) &= \frac{(x^2+y^2)\cdot 1 - x\cdot 2x}{(x^2+y^2)^2} = \cancel{x^2+y^2}\ \frac{y^2-x^2}{(x^2+y^2)^2} \\ \frac{\partial}{\partial y}\left(\frac{-y}{x^2+y^2}\right) &= \frac{-(x^2+y^2)+y\cdot 2y}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2} \end{aligned}\right\} \text{equal!}$$

(2) integrate around unit circle



$$\int_C \underline{F}\cdot d\underline{s} \quad \text{unit circle} \quad c(\theta) = \langle \cos\theta, \sin\theta \rangle \quad 0 \le \theta \le 2\pi$$

$$c'(\theta) = \langle -\sin\theta, \cos\theta \rangle.$$

$$= \int_0^{2\pi} \underline{F}(s(\theta))\cdot \underline{c}'(\theta)\, d\theta$$

$$= \int_0^{2\pi} \left\langle \frac{-\sin\theta}{1}, \frac{\cos\theta}{1} \right\rangle \cdot \langle -\sin\theta, \cos\theta \rangle\, d\theta = \int_0^{2\pi} 1\, d\theta = 2\pi \neq 0!$$

(3) find potential function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

check $\nabla f = \left\langle \frac{1}{1+(y/x)^2}\cdot\frac{-y}{x^2}, \frac{1}{1+(y/x)^2}\cdot\frac{1}{x} \right\rangle = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \right\rangle$

(4) $\tan^{-1}\left(\frac{y}{x}\right)$ not defined at $(0,0)$ so $D = \mathbb{R}^2 \setminus (0,0)$ not simply connected

(in polars $\tan^{-1}\left(\frac{y}{x}\right) = \theta$)

(5) fact. $\oint_C F\cdot ds = 2\pi n$ n = winding number