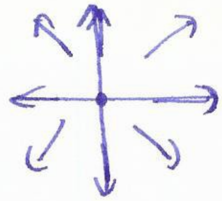
② $\underline{F}(x,y) = \langle x, y \rangle$



③ any gradient vector field $\underline{F} = \nabla f \quad f: \mathbb{R}^2 \to \mathbb{R}$

<u>Def</u>: A vector field is <u>conservative</u> if $\underline{F} = \nabla f$ for some scalar function f called the <u>potential function</u> for $\underline{F}$.

<u>Example</u> (3d) (inverse square field) $\underline{F}(x,y,z) = \underline{F}(\underline{x}) = \frac{1}{\|x\|^3} \underline{x}$

(<u>note</u>: $\|\underline{F}(x)\| = \frac{1}{\|\underline{x}\|^2}$).

<u>claim</u>: this is conservative, with potential function $f(x) = \frac{1}{\|\underline{x}\|} = \frac{-1}{\sqrt{x^2+y^2+z^2}}$

<u>check</u>: $\nabla f = \langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \rangle = \langle +\frac{1}{2}(x^2+y^2+z^2)^{-3/2} \cdot 2x, \dots \rangle$

$$= \frac{\langle x, y, z \rangle}{(\sqrt{x^2+y^2+z^2})^3} = \frac{\underline{x}}{\|\underline{x}\|^3}.$$

<u>Warning</u>: not all vector fields are conservative.

<u>Q</u>: how do you know if $\underline{F}$ is conservative?

<u>2d</u>: <u>Thm</u> $\underline{F}(x,y) = \langle A(x,y), B(x,y) \rangle$ is conservative iff $\frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$.

<u>note</u> $\nabla f = \langle \underbrace{\frac{\partial f}{\partial x}}_{A}, \underbrace{\frac{\partial f}{\partial y}}_{B} \rangle$ so $\frac{\partial A}{\partial y} = f_{xy}$ $\frac{\partial B}{\partial x} = f_{yx}$. and mixed partials are equal so $\underline{F} = \nabla f \Rightarrow \frac{\partial B}{\partial x} = \frac{\partial A}{\partial y}$.

<u>3d</u>: <u>Def</u> curl $\underline{F} = \nabla \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A & B & C \end{vmatrix} = \langle C_y - B_z, -C_x + A_z, B_x - A_y \rangle$

$\underline{F} = \langle A, B, C \rangle$

<u>Thm</u> $\underline{F}$ is conservative iff $\nabla \times \underline{F} = \underline{0}$

Finding potential functions

example (2d) $\underline{F}(x,y) = \langle y, x \rangle = \nabla f$ for some f.

$$\left.\begin{aligned} \frac{\partial f}{\partial x} = y &\Rightarrow f(x,y) = xy + c_1(y) \\ \frac{\partial f}{\partial y} = x &\Rightarrow f(x,y) = xy + c_2(x) \end{aligned}\right\} \text{ but } \begin{aligned} \frac{\partial}{\partial y} c_1 = 0 \\ \frac{\partial c_2}{\partial x} = 0 \end{aligned}$$

$$\Rightarrow f(x,y) = xy + c$$

c constant.

Other vector operators

Divergence 2d : $\underline{F}(x,y) = \langle A, B \rangle$ $\quad \underline{\nabla}.\underline{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y}$ (scalar!)

3d : $F(x,y,z) = \langle A, B, C \rangle$ $\quad \underline{\nabla}.\underline{F} = \frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}$ (scalar!)

(interpretation: $\underline{\nabla}.\underline{F} = 0 \leftrightarrow$ incompressible fluid flow).

useful fact $\underline{\nabla}.\underline{\nabla} \times \underline{F} = 0$

note

$$\text{function} \xrightarrow[\text{grad}]{\nabla} \text{vector fields} \xrightarrow[\text{curl}]{\underline{\nabla}\times} \text{vector fields} \xrightarrow[\text{div}]{\underline{\nabla}.} \text{function}$$

$$f \longmapsto \nabla f$$

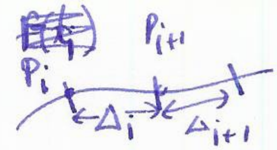
$$\underline{F} \longmapsto \underline{\nabla} \times F$$

$$\underline{F} \longmapsto \underline{\nabla}.\underline{F}.$$

§16.2 Line integrals

§ 16.2 Line integrals

parameterized curve C : 



$\underline{c}(t)$

scalar line integral : $\int_C f(x,y,z)\,ds$ ($f(x,y,z)$ scalar function $f: \mathbb{R}^3 \to \mathbb{R}$)

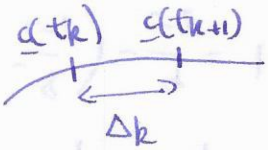
can define as Riemann sum $\int_C f(x,y,z)\,ds = \lim_{|\Delta s_i| \to 0} \sum_{i=1}^{n} f(P_i)\Delta_i$

Thm: $\underline{c}(t)$ $a \le t \le b$ parameterized curve C

then $\int_C f(x,y,z)\,ds = \int_a^b f(\underline{c}(t))\, \|\underline{c}'(t)\|\, dt$

and this does not depend on the choice of parameterization!

Note if ~~dt~~ $f(x,y,z) = 1$ then get $\int_a^b \|\underline{c}'(t)\|\,dt$ = arc length of curve.

intuition: 

length of this segment is $\int_{t_k}^{t_{k+1}} \|c'(t)\|\,dt$

so want $\lim_{n\to\infty} \sum f(t_k) \int_{t_k}^{t_{k+1}} \|c'(t)\|\,dt \sim \int_a^b f(t)\,\|c'(t)\|\,dt$

Notation $ds = \|c'(t)\|\,dt = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$

Example find $\int_C (x+y+z)\,ds$ where C is the helix

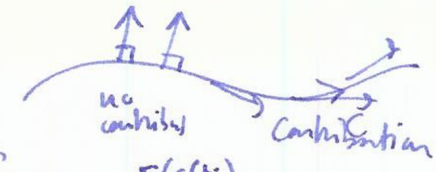
$\underline{c}(t) = (\cos t, \sin t, t)$ for $0 \le t \le 2\pi$

$\underline{c}'(t) = (-\sin t, \cos t, 1)$

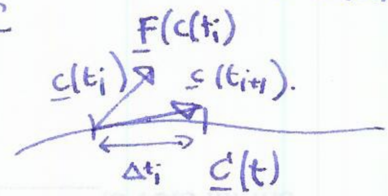
$= \int_0^{2\pi} (\cos t + \sin t + t)\sqrt{\sin^2 t + \cos^2 t + 1}\,dt = 2\int_0^{2\pi} \cos t + \sin t + t\,dt$

$= 2\left[\sin t - \cos t + \frac{1}{2}t^2\right]_0^{2\pi} = \frac{1}{2}(2\pi)^2 = 2\pi^2.$

<u>vector line integrals</u> notation $\int_C \underline{F} \cdot \underline{ds}$



intuition: integral of tangential component of $\underline{F}$ along C



Riemann sum def$^{\underline{n}}$: $\sum_i F(\underset{c(t_i)}{\cancel{x_{i+1}}}) \cdot c'(t)\,\Delta t_i$

<u>Def$^{\underline{n}}$</u> Let $T = \dfrac{\underline{c}'(t)}{\|\underline{c}'(t)\|}$ unit tangent vector to $\underline{c}$

then $\int_C \underline{F} \cdot ds = \int (\underline{F} \cdot \underline{T})\, ds$

If $\underline{c}$ has parameterization $\underline{c}(t)$ for $a \le t \le b$:

$$\int_C \underline{F} \cdot \underline{ds} = \int_a^b \underline{F}(\underline{c}(t)) \cdot \frac{c'(t)}{\|\underline{c}(t)\|} \|\underline{c}'(t)\|\, dt = \int_a^b F(\underline{c}(t)) \cdot c'(t)\, dt.$$

<u>summary</u>: $\boxed{\int_C \underline{F} \cdot \underline{ds} = \int_a^b \underline{F}(\underline{c}(t))\, dt}$

<u>Fact</u>: this does not depend on the parameterization.

<u>Alternate notation</u>: $\int_C \underline{F} \cdot \underline{ds}$ $\quad \underline{F} = \langle F_1, F_2, F_3 \rangle$

then write $\int_C \underline{F} \cdot \underline{ds} = \int_C F_1\, dx + F_2\, dy + F_3\, dz$

in terms of $\underline{c}(t)$ $= \int_a^b F_1(\underline{c}(t)) \frac{dx}{dt}\, dt + F_2(\underline{c}(t)) \frac{dy}{dt} + F_3(\underline{c}(t)) \frac{dz}{dt}$

(same as before).

<u>Example</u> (2d) Integrate $\underline{F} = \langle 2y, -3 \rangle$ over the ellipse $\underline{c}(t) = (4 + 3\cos\theta,\ 3 + 2\sin\theta)$
$0 \le \theta \le 2\pi$.