## §15.5 Change of variable



**1d** $\displaystyle\int_a^b f(x)\,dx \quad x(u) \quad \longleftrightarrow \quad \int_c^d f(x(u))\,\frac{dx}{du}\,du \qquad x(d)=b, \quad x(c)=a$

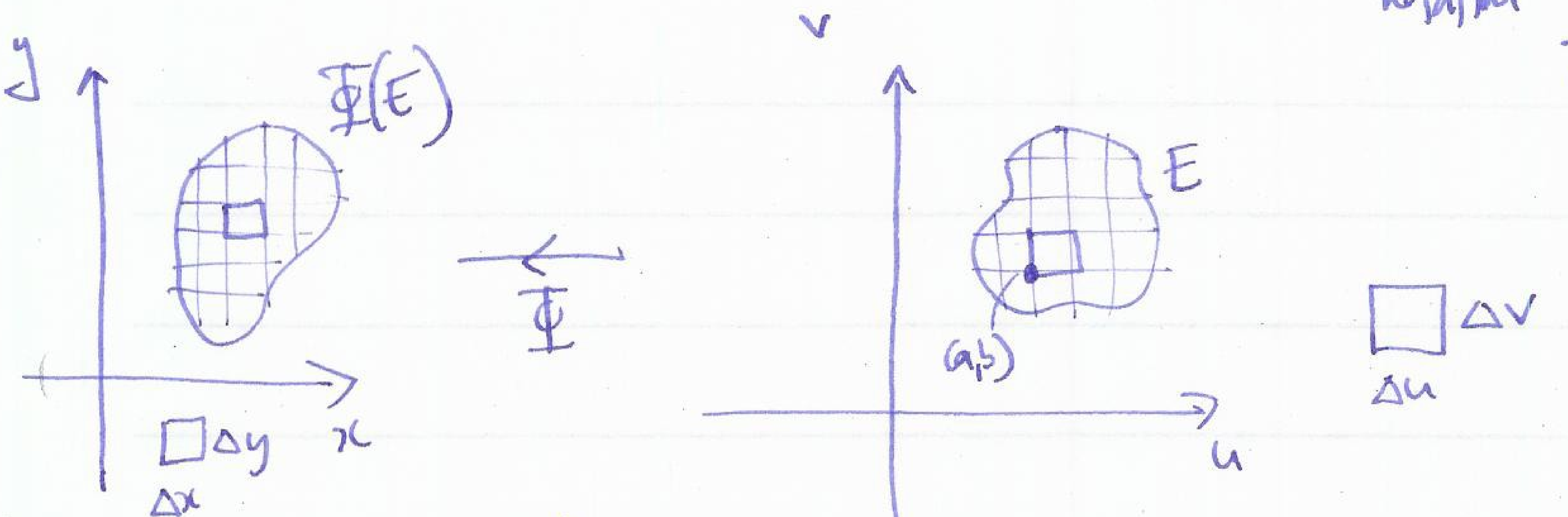
**2d** $\displaystyle\iint_D f(x,y)\,dA \qquad \begin{matrix} x(u,v) \\ y(u,v)\end{matrix} \longleftrightarrow \iint_D f(x(u,v),y(u,v))\; J\,du\,dv$

$D$: limits in terms of $x,y$; $dA$: $dx\,dy$ or $dy\,dx$

$D$: in terms of $u,v$

$$J = \left|\, \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \,\right|.$$

notation $\quad J = \left|\, \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| \,\right|.$




$$(x(u,v),\, y(u,v)) \xleftrightarrow{\;\phi\;} (u,v)$$


linear approximation: $\quad (u,v) \xmapsto{\;\Phi\;} (x(u,v),\, y(u,v))$

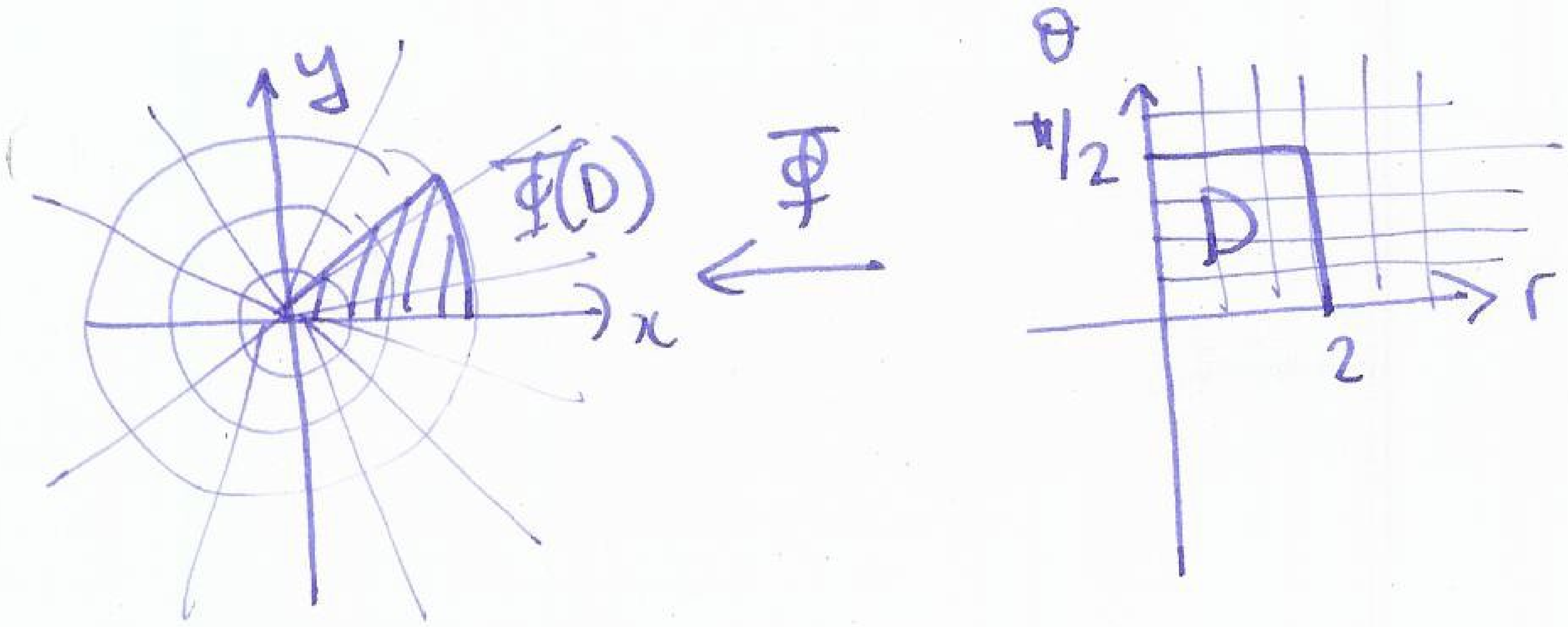
$$\left( x(a,b) + \frac{\partial x}{\partial u}(a,b)(u-a) + \frac{\partial x}{\partial v}(a,b)(v-b),\;\; y(a,b) + \frac{\partial y}{\partial u}(a,b)(u-a) + \frac{\partial y}{\partial v}(a,b)(v-b) \right)$$

$$= (x(a,b),\, y(a,b)) + \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \begin{bmatrix} u-a \\ v-b \end{bmatrix}$$

need to replace

$$\Delta x \Delta y \quad \text{with} \quad \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \Delta u \Delta v \quad \text{or} \quad dx\,dy = J(\Phi)\,du\,dv$$

Example polar coordinates




$$\overset{x}{\phantom{.}}\quad\overset{y}{\phantom{.}}$$
$$\Phi(r,\theta) = (r\cos\theta, r\sin\theta)$$

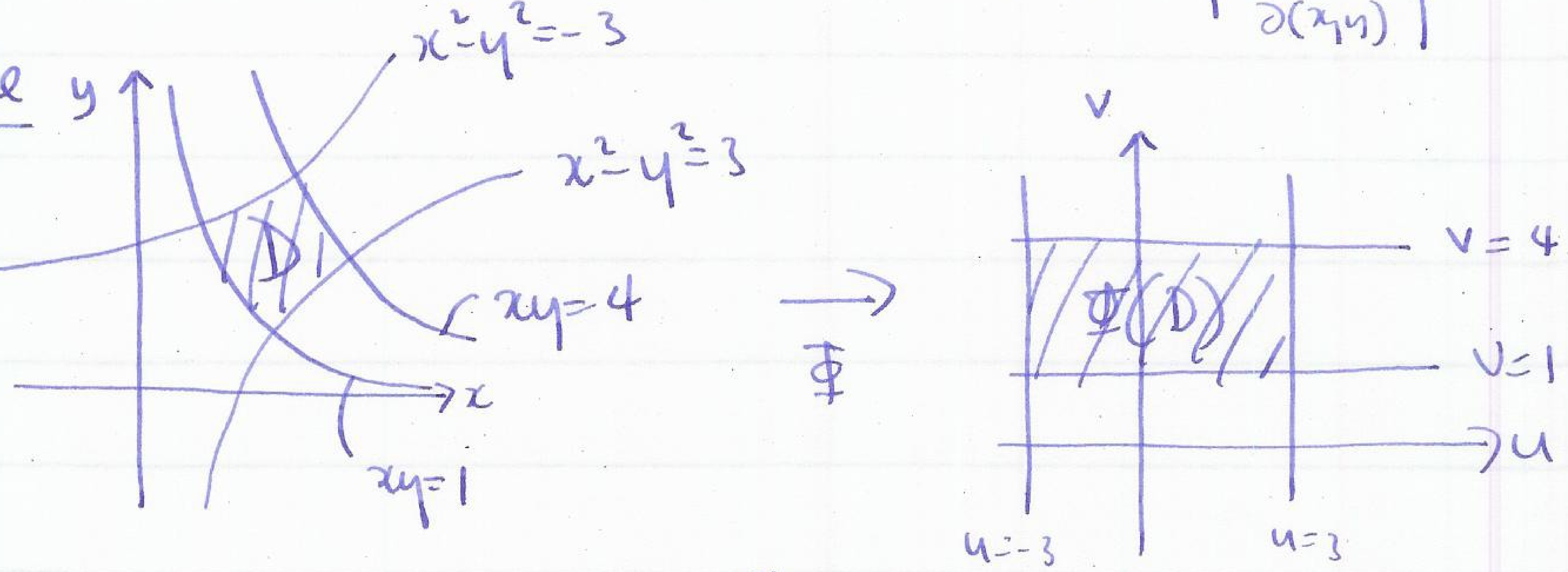
find $J(\Phi)$

$$J(\Phi) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$$

so $dx\,dy = r\,dr\,d\theta$

Useful fact $\quad J(\Phi) = \dfrac{1}{J(\Phi^{-1})} \iff \left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \dfrac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|}$

Example




$$(x,y) \longmapsto (\overset{u}{x^2-y^2}, \overset{v}{xy})$$

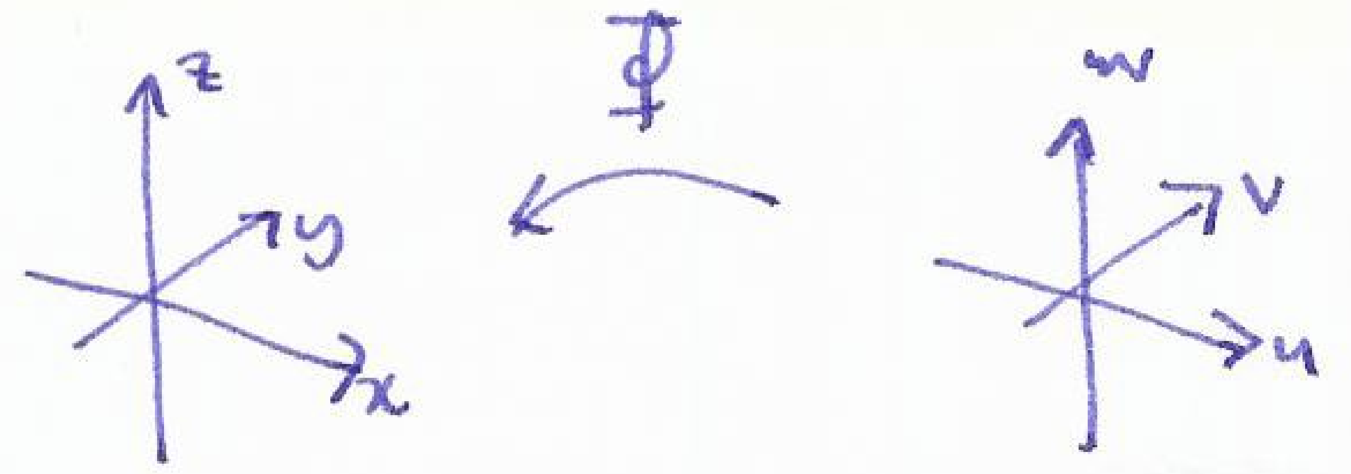
$$dx\,dy = \underbrace{\left|\frac{\partial(x,y)}{\partial(u,v)}\right|}_{\frac{1}{2x^2+2y^2}} du\,dv \qquad \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2x^2+2y^2$$

(usually want this in terms of $u,v$: $\quad x^2+y^2 = x^2-y^2+2y^2 = u+2y^2$

$v = xy \Rightarrow x = \frac{v}{y}$ so $\frac{v^2}{u^2} - u^2 = u \qquad u^4 + uy^2 - v^2 = 0 \qquad y = \frac{-u \pm \sqrt{u^2+4v^2}}{2}$

so $x^2+y^2 = u + 2y^2 = \sqrt{u^2+4v^2}$.

3 variables




$$dx\,dy\,dz = |J|\,du\,dv\,dw$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right|$$

Example: spherical coordinates

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

$$J = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\phi \sin\theta & \rho \cos\theta \cos\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \\ \cos\phi & 0 & -\rho \sin\phi \end{vmatrix}$$

$$= \left| \cos\phi \begin{vmatrix} -\rho \sin\theta \sin\phi & \rho \cos\theta \cos\phi \\ \rho \cos\theta \sin\phi & \rho \sin\theta \cos\phi \end{vmatrix} - \rho \sin\phi \begin{vmatrix} \cos\theta \sin\phi & -\rho \sin\theta \sin\phi \\ \sin\theta \sin\phi & \rho \cos\theta \sin\phi \end{vmatrix} \right|$$

$$= \left| \rho^2 \left[ \cos\phi \left( -\sin^2\theta \sin\phi \cos\phi - \cos^2\theta \sin\phi \cos\phi \right) - \sin\phi \left( \cos^2\theta \sin^2\phi + \sin^2\theta \sin^2\phi \right) \right] \right|$$

$$= \left| \rho^2 \left[ -\sin\phi \cos^2\phi \underbrace{\left( \sin^2\theta + \cos^2\theta \right)}_{1} - \sin^3\phi \underbrace{\left( \cos^2\theta + \sin^2\theta \right)}_{1} \right] \right|$$

$$= \rho^2 \sin\phi \left( \sin^2\phi + \cos^2\phi \right) = \rho^2 \sin\phi$$

## §16.1 Vector fields

Def^n A vector field is a function which assigns a vector to each point in the domain.

Examples (2d) ① $\underline{F}(x,y) = \langle 1, 0 \rangle$