$$\int_{-\pi/2}^{\pi/2} \frac{4}{3}\left(4-4\sin^2\theta\right)^{3/2}\cdot 2\cos\theta \, d\theta = \frac{8}{3}\int_{-\pi/2}^{\pi/2} 4^{3/2}\cos^3\theta\cos\theta \, d\theta$$

$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \cos^4\theta \, d\theta \qquad \begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1\\ \cos^2\theta &= \tfrac{1}{2}\cos 2\theta + \tfrac{1}{2}\end{aligned}$$

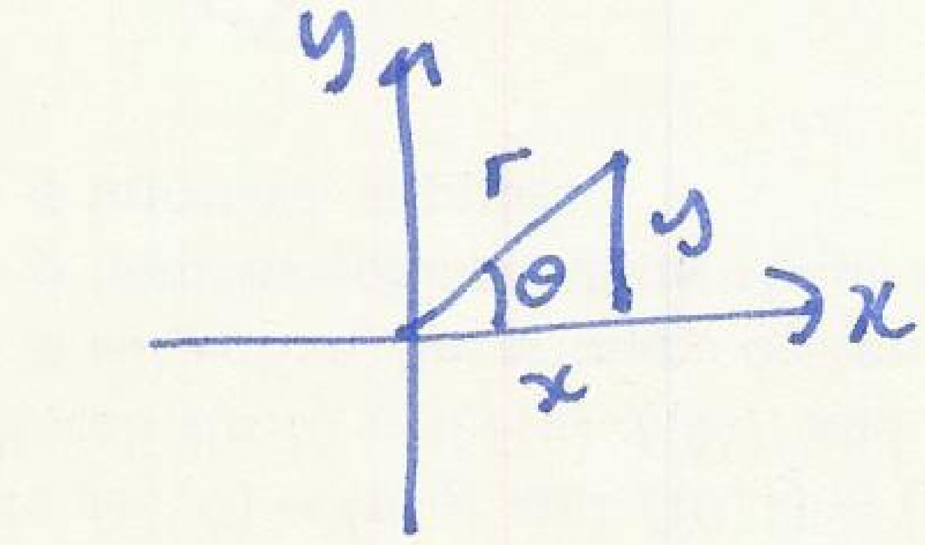
$$= \frac{64}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{4}\left(\cos^2 2\theta + 2\cos 2\theta + 1\right) d\theta$$

$$= \frac{16}{3}\int_{-\pi/2}^{\pi/2} \frac{1}{2}\cos 4\theta + \frac{1}{2} + 2\cos 2\theta + 1 \, d\theta$$

$$= \frac{16}{3}\left[-\frac{1}{8}\sin 4\theta + \frac{3\theta}{2} - \sin 2\theta\right]_{-\pi/2}^{\pi/2} = \frac{16}{3}\cdot\frac{3}{2}\cdot 2\frac{\pi}{2} = 8\pi.$$

§12.7 Cylindrical and spherical coordinates

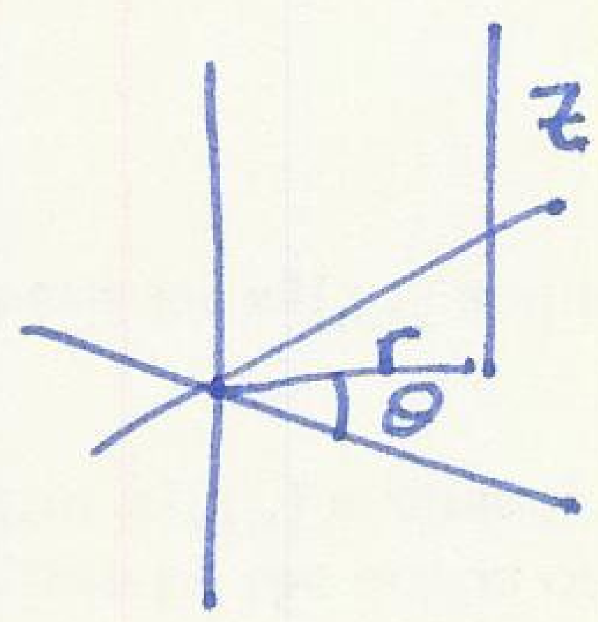
Polar coords:



$r^2 = x^2+y^2 \qquad x = r\cos\theta$

$\tan\theta = y/x \qquad y = r\sin\theta$

Cylindrical coords:

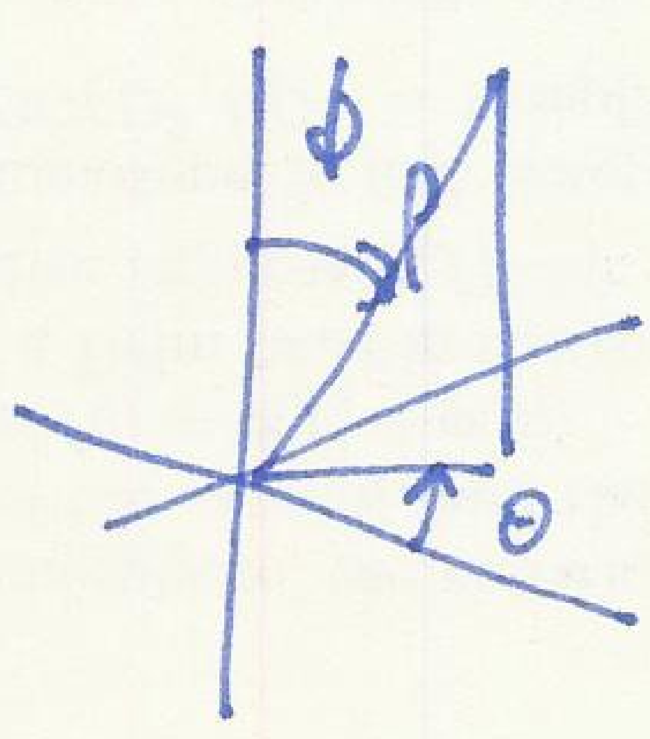


$x = r\cos\theta \qquad r = \sqrt{x^2+y^2}$

$y = r\sin\theta \qquad \theta = \tan^{-1}(y/x)$

$z = z \qquad z = z$

Spherical coords:



$x = \rho\cos\theta\sin\phi \qquad \rho^2 = x^2+y^2+z^2$

$y = \rho\sin\theta\sin\phi \qquad \theta = \tan^{-1}(y/x)$

$z = \rho\cos\phi \qquad \phi = \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$

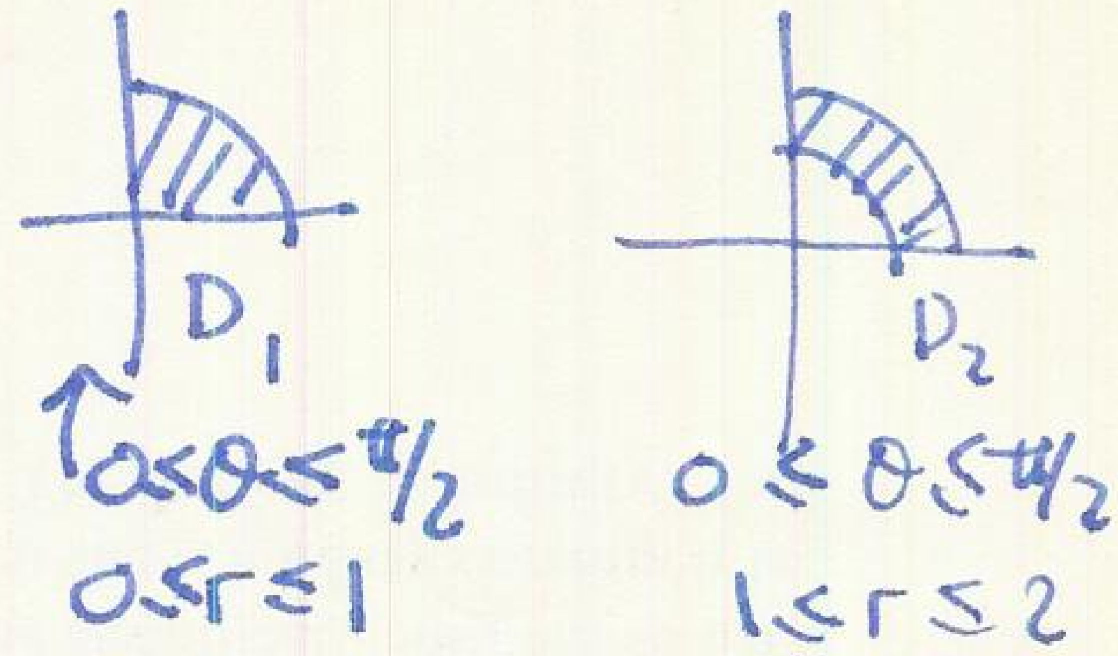
some sets are easier to describe in these coordinates.

 $x^2+y^2=1 \Leftrightarrow r=1$ in cylindricals

 $x^2+y^2+z^2=1 \Leftrightarrow \rho=1$ in sphericals

§15.4 Integration in polar, cylindrical and spherical coordinates

recall: double integrals over regions $\iint_D f(x,y)\,dA$



some regions easier to describe in polars.

useful fact: in polar coordinates

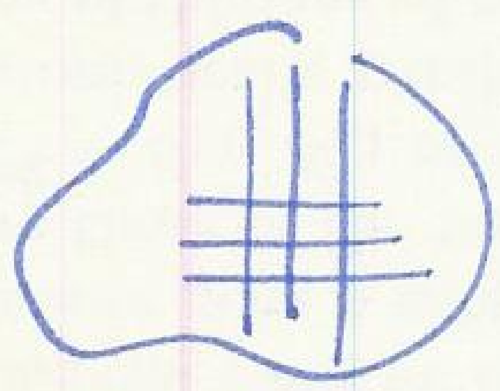
$$\iint_D f(x,y)\,dA = \iint_D \underset{\substack{\uparrow \\ f(x(r,\theta),\, y(r,\theta))}}{f(r,\theta)}\, r\,dr\,d\theta$$

i.e. $dA = dx\,dy$ in cartesians

$\boxed{dA = r\,dr\,d\theta}$ in polars

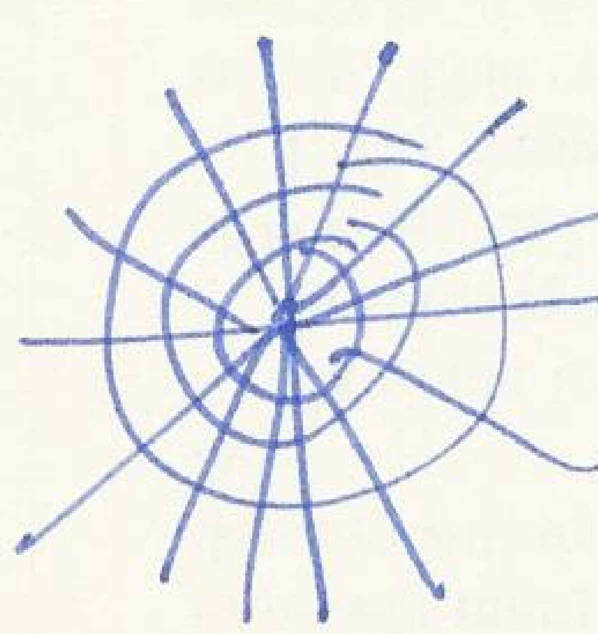
Warning: do not forget the r.

idea: cartesians



Δx, Δy area $\Delta A = \Delta x \Delta y$

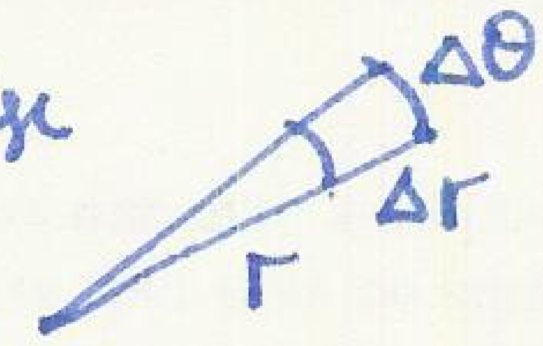
polars:



"squares" are different sizes!

area $\Delta A \approx r \Delta r \Delta \theta$

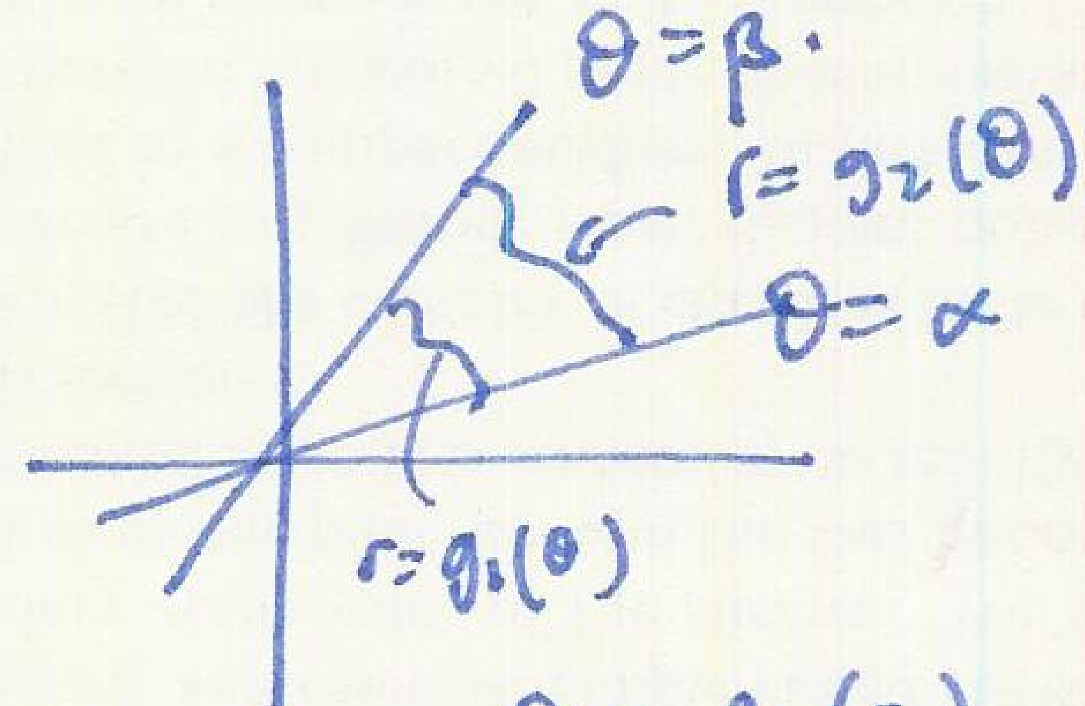
area of wedge



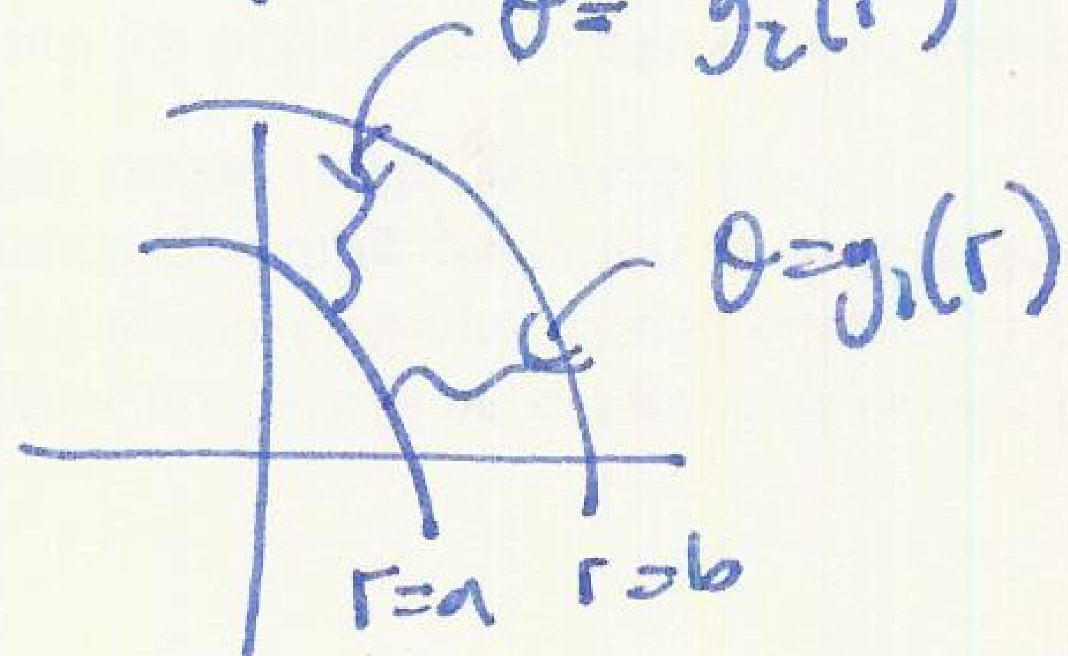
area is $\approx (r+\Delta r)\Delta\theta - r\Delta\theta = r\Delta r\Delta\theta$

Describing regions

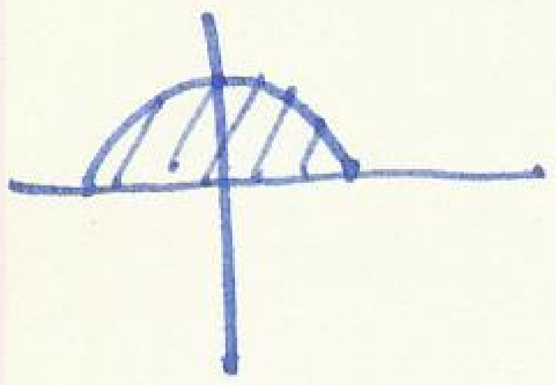
$$\int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)\, r\,dr\,d\theta$$



$$\int_a^b \int_{g_1(r)}^{g_2(r)} f(r,\theta)\, r\,d\theta\,dr$$



<u>Example</u> Integrate $f(x,y) = xy$ over the upper half disc of radius 4



$0 \le \theta \le \pi$
$1 \le r \le 4$

$$\iint_D f(x,y)\, dA \qquad \int_0^{\pi} \int_0^4 r^2 \cos\theta \sin\theta \; r\, dr\, d\theta$$

$$= \int_0^{\pi} \tfrac{1}{2}\sin 2\theta \underbrace{\int_0^4 r^3\, dr}_{\left[\frac{1}{4}r^4\right]_0^4 = 4^3} d\theta = 32\left[-\tfrac{1}{2}\cos 2\theta\right]_{\pi}^{0} = 0$$

<u>Cylindrical coords</u> $\iiint_R f(x,y,z)\, dV \qquad \boxed{dV = r\, dr\, d\theta\, dz}$

<u>spherical coords</u> $\boxed{dV = \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi}$