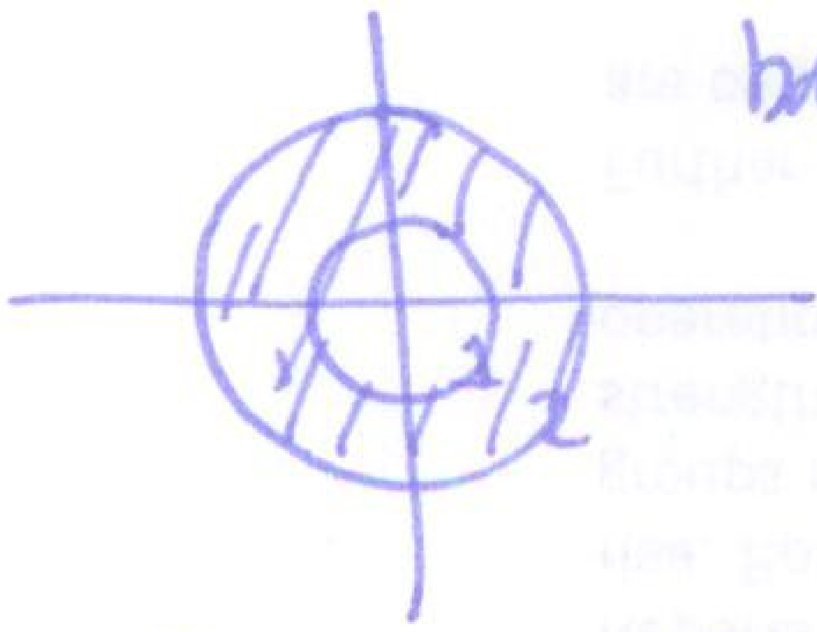
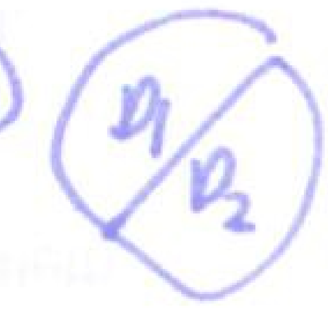
§15.2 Double integrals over general regions

Problem. boundaries of region not functions of x or y.

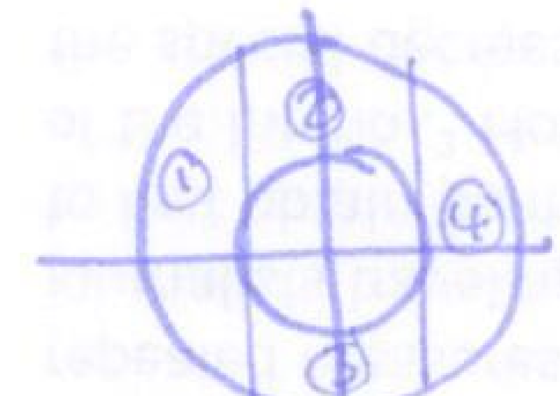


observation if a region D is the union of two subregions D_1, D_2 then

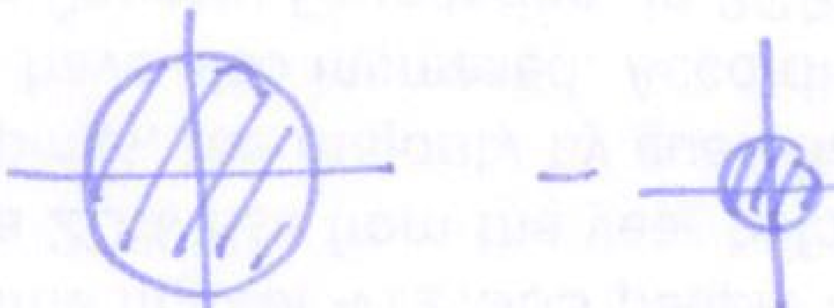


$$\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA$$

Solution ① chop region into nice regions

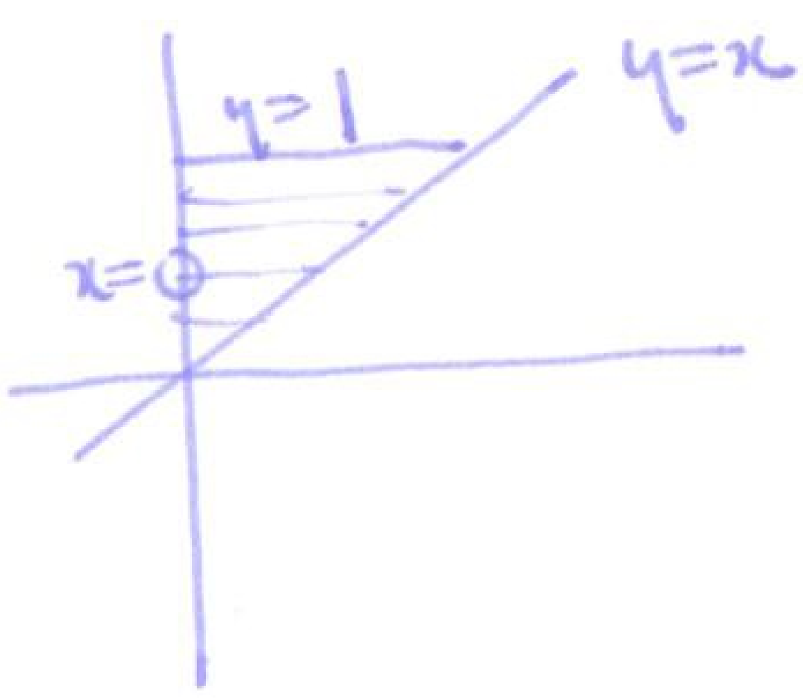


② write D as $D_1 \setminus D_2$

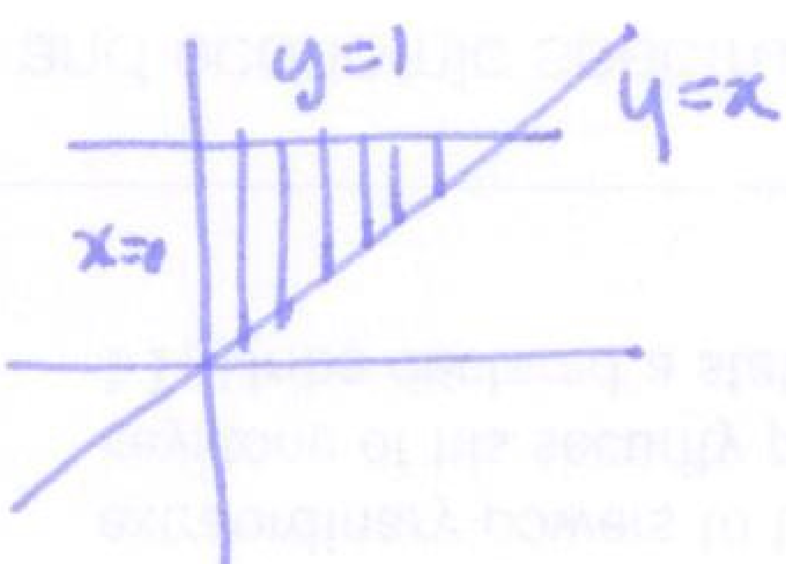


$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} f(x,y)\,dy\,dx \qquad - \qquad \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y)\,dy\,dx$$

Changing order of integration

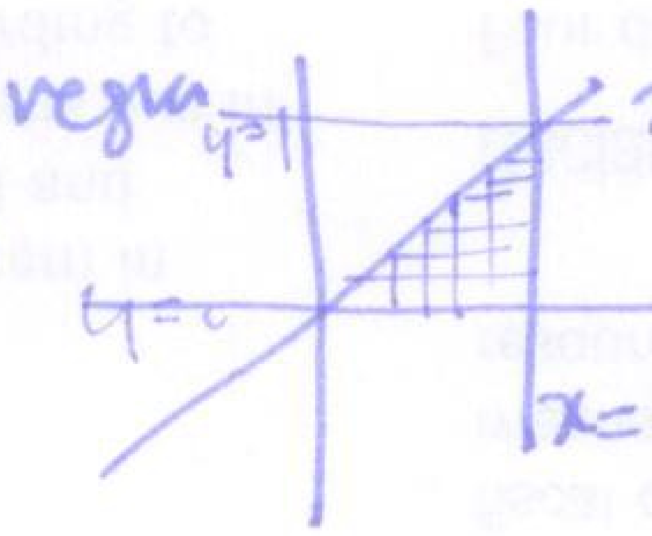


$$\int_0^1\int_0^y f(x,y)\,dx\,dy$$



$$\int_0^1\int_x^1 f(x,y)\,dy\,dx$$

Example switch limits: $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$ draw region



new limits $\displaystyle\int_0^1\int_0^x \frac{\sin x}{x}\,dy\,dx$

can now do integral...

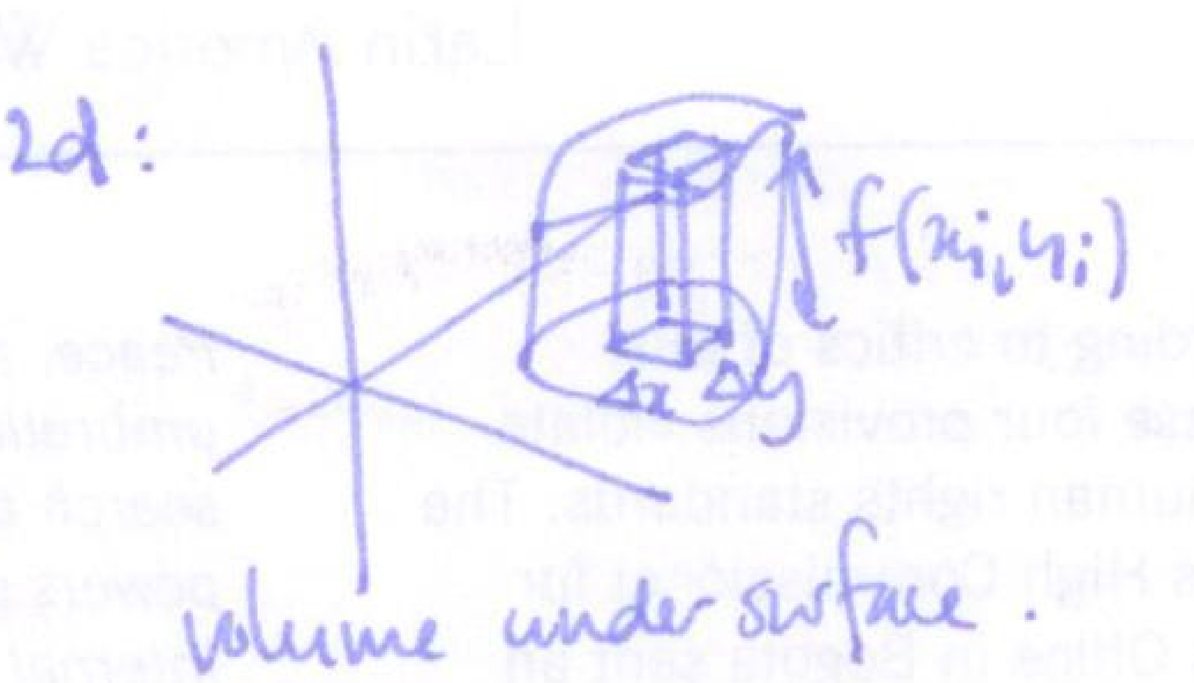
§15.3 Triple integrals

1d:

area under curve

$\sum f(x_i)\Delta x$

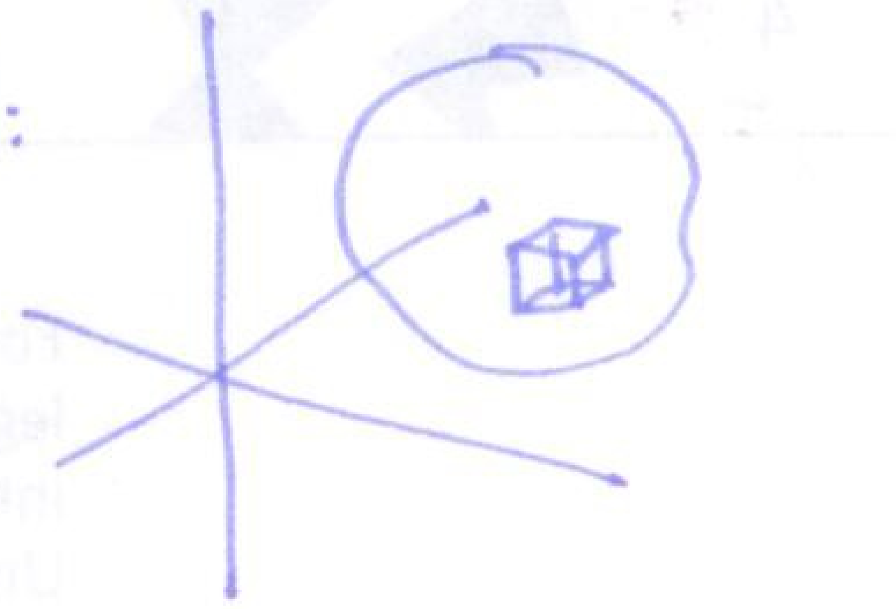
2d:



volume under surface

$\sum f(x_i, y_i)\Delta x \Delta y$

3d:



$\sum f(x_i, y_i, z_i)\Delta x \Delta y \Delta z$

notation $\displaystyle\iiint_R f(x,y,z)\,dV$

Example integrate $f(x,y,z) = x^2 e^{y+3z}$ over $\underset{x}{[1,2]} \times \underset{y}{[3,4]} \times \underset{z}{[0,1]}$

$$\underbrace{\int_0^1 \int_3^4 \int_1^2 x^2 e^{y+3z}\,dx\,dy\,dz}$$

$$\text{inner integral} = e^{y+3z}\int_1^2 x^2\,dx = e^{y+3z}\left[\tfrac{1}{3}x^3\right]_1^2 = \frac{7}{3}e^{y+3z}$$

$$\underbrace{\int_0^1 \int_3^4 \frac{7}{3}e^{y+3z}\,dy\,dz} = \frac{7}{3}e^{3z}\int_3^4 e^y\,dy = \frac{7}{3}e^{3z}\left[e^y\right]_3^4 = \frac{7}{3}e^{3z}(e^4 - e^3)$$

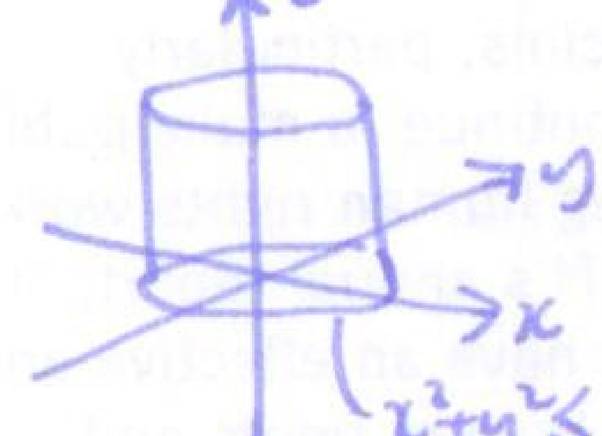
$$\int_0^1 \frac{7}{3}e^{3z}(e^4-e^3)\,dz = \frac{7}{3}(e^4-e^3)\int_0^1 e^{3z}\,dz = \frac{7}{3}(e^4-e^3)\left[\frac{1}{3}e^{3z}\right]_0^1$$

$$= \frac{7}{3}(e^4-e^3)\frac{1}{3}(e^3-1).$$

difficult bit: describing regions

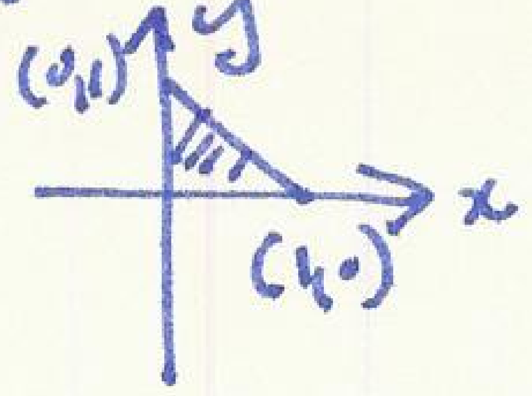
Region $R \Longleftrightarrow$ described by limits to $dx\,dy\,dz$ in same order.

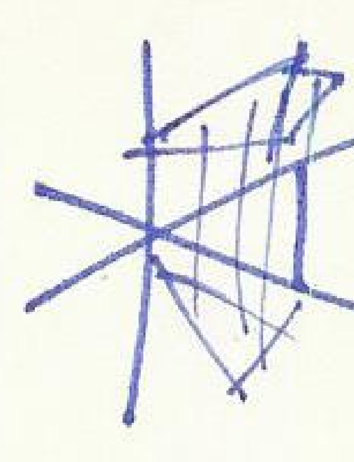
Example.



$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}\int_0^1 f(x,y,z)\,dz\,dy\,dx$$

<u>Example</u> $R=$ region between planes $z=x+y$ and $z=3x+5y$ over the triangle





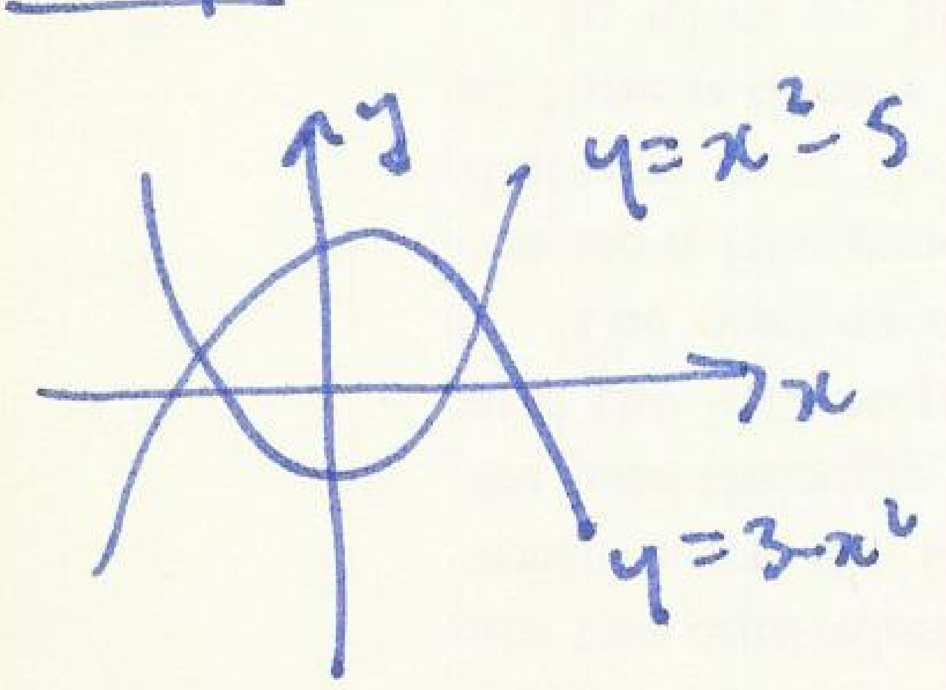
$$\int_0^1 \int_0^{1-x} \int_{x+y}^{3x+5} f(x,y,z)\, dz\, dy\, dx$$

$R=$ sphere

$x^2+y^2+z^2 \leq 1$

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} f(x,y,z)\, dz\, dy\, dx$$

<u>Example</u> (20) find the area of the region between $y=3-x^2$ and $y=x^2-5$



find intersections: $3-x^2 = x^2-5$

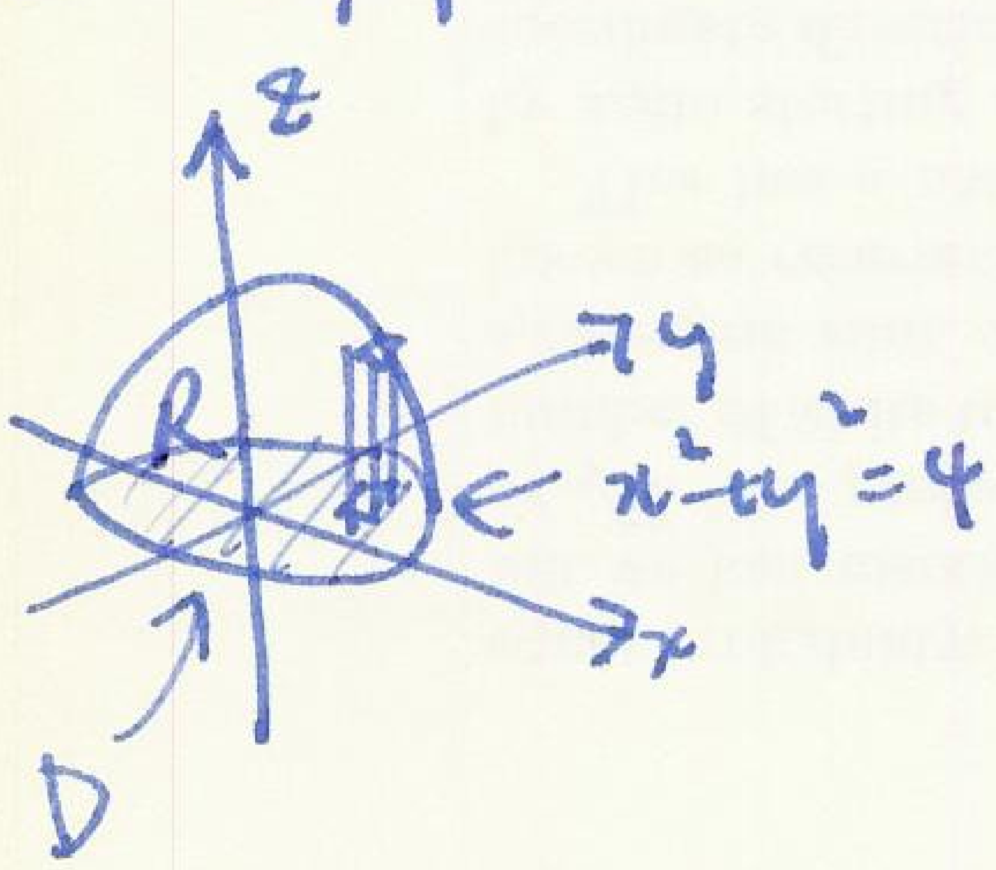
$2x^2 = 8 \quad x = \pm 2.$

$$\int_{-2}^{+2} \int_{x^2-5}^{3-x^2} 1\, dy\, dx \qquad = \Big[y\Big]_{x^2-5}^{3-x^2}$$

$$= 8-2x^2.$$

$$\int_{-2}^{2} 8-2x^2\, dx = \Big[8x - \tfrac{2}{3}x^3\Big]_{-2}^{2} = 16 - \frac{16}{3} + 16 - \frac{16}{3}.$$

<u>Example</u> find the volume of the region below $z=4-x^2-y^2$ and above the xy-plane: can do this in 2 ways



$$\iiint_R 1\, dV = \text{vol} \qquad \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} \int_0^{4-x^2-y^2} 1\, dz\, dy\, dx$$

$$\iint_D 4-x^2-y^2\, dA \qquad \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{+\sqrt{4-x^2}} 4-x^2-y^2\, dy\, dx.$$

$$\int_{-2}^{2} \frac{4}{3}(4-x^2)^{3/2}\, dx \qquad \text{try } x = 2\sin\theta$$

$$\frac{dx}{d\theta} = 2\cos\theta$$