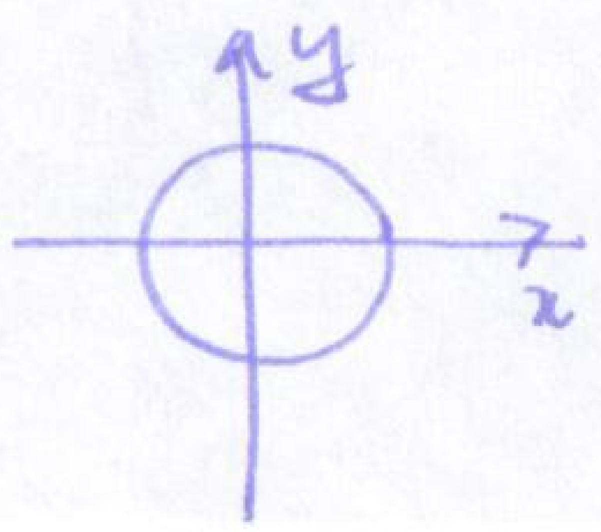
<u>Example</u> find the abs max/min of $f(x,y) = xy$ on the unit disc $x^2+y^2 \le 1$



① find critical points

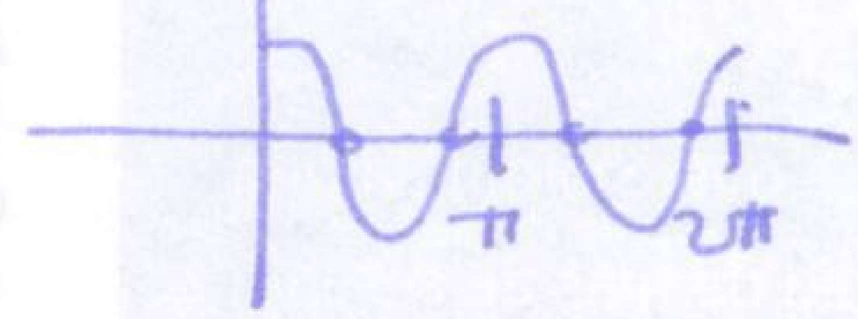
$$\left.\begin{array}{l} f_x = y \\ f_y = x \end{array}\right\} \quad (0,0) \text{ only critical point}$$

② find max/min on boundary

parameterize boundary $(\cos\theta, \sin\theta)$ $\quad 0 \le \theta \le 2\pi$.

$$f(\theta) = f(x(\theta), y(\theta)) = \cos\theta\sin\theta = \tfrac{1}{2}\sin 2\theta$$

$$f'(\theta) = \cos 2\theta \qquad f'(\theta) = 0 \qquad \theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$$



corresponds to points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$ $\quad f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \pm\frac{1}{2}$.

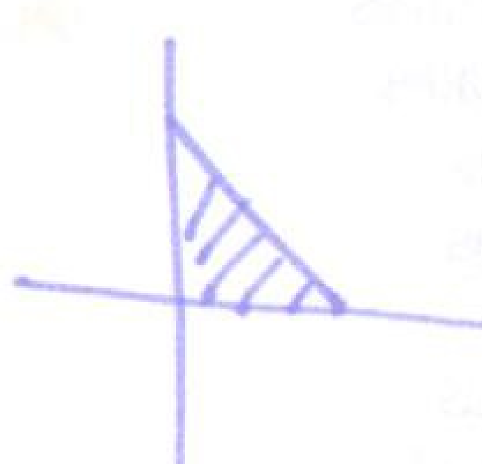
<u>Example</u> find the box of max volume bounded by the coordinate planes and the plane $x+y+z=1$

sides of box x, y, z, with $x+y+z=1$

volume of box $V = xyz$

so sides $x, y, \; z = 1-x-y$

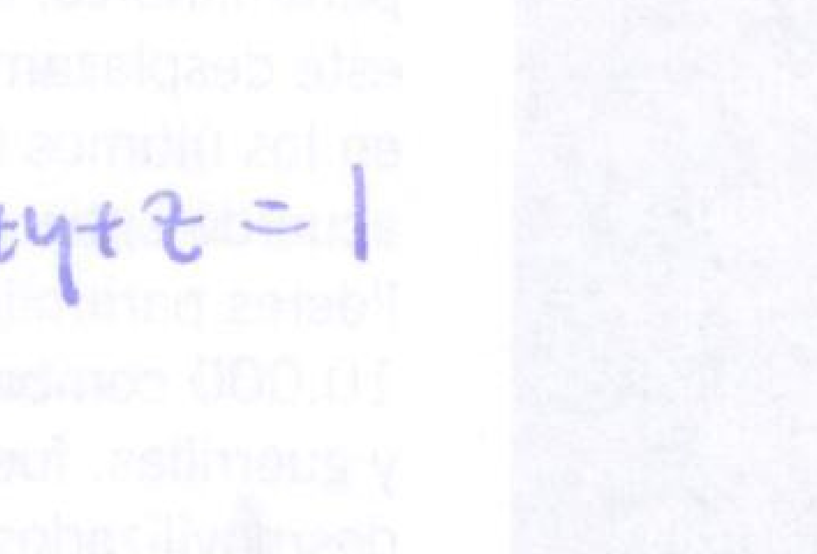
$V = xy(1-x-y) \leftarrow$ find max of this on $\begin{array}{l} x \ge 0 \\ y \ge 0 \\ z \ge 0 \Leftrightarrow 1-x-y \ge 0 \\ x+y \le 1. \end{array}$



① find critical points $\quad V_x = y - 2xy - y^2$

$\qquad V_y = x - x^2 - 2xy$

$$\text{solve } \left.\begin{array}{l} V_x = 0 \\ V_y = 0 \end{array}\right\} \begin{array}{l} ①: \; y(1-2x-y) = 0 \\ ②: \; x(1-x-2y) = 0 \end{array}$$



($x=0$ or $y=0$ on boundary)

$$\text{lines intersect at } \left.\begin{array}{l} 2x+y=1 \\ x+2y=1 \end{array}\right\} \quad ① - 2②: \; -3y = -1 \quad y = 1/3 \quad x = 1/3.$$

$(1/3, 1/3)$ critical point.

② check max on boundary

$x=0 \quad 0\le y\le 1 \quad V=0$

$y=0 \quad 0\le x\le 1 \quad V=0$

$(t,t-1) \quad 0\le t\le 1 \quad V = t(1-t)\big(1-t-(1-t)\big) = 0.$

$\Big\}$ $V=0$ on boundary

so max at $V(1/3, 1/3) = \frac{1}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{1}{27}$.

§14.8 Lagrange multipliers

optimization with constraint i.e. max $f(x,y)$ subject to $g(x,y)=0$

Example maximize $f(x,y) = \sqrt{x^2+y^2}$ subject to $\underbrace{x+y-4}_{g(x,y).} = 0$

$g(x,y)=0$

consider level sets $f(x,y)=c$

key point: at the extreme values of $f(x,y)$ on $g(x,y)=0$, the level sets of f are parallel to g

$g(x,y)=0$ so we want to look for points, where level sets of f parallel to levels sets of g

i.e. solve $\nabla f = \lambda \nabla g$

Example $\nabla f = \left\langle \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x,\ \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y \right\rangle = \lambda \nabla g = \lambda\langle 1,1\rangle$ $(\lambda \ne 0)$

$$\left.\begin{aligned} \frac{x}{\sqrt{x^2+y^2}} &= \lambda \\ \frac{y}{\sqrt{x^2+y^2}} &= \lambda \end{aligned}\right\} \quad \frac{x}{y} = 1 \Rightarrow x = y$$

now plug in to $g(x,y)=0$ $\quad x+y-4=0$

$x+x-4=0 \Rightarrow x=2$

$y=2$

<u>summary</u> want to max/min f given $g=0$

solve $\nabla f = \lambda \nabla g$, $g=0$.

<u>Example</u> find the point in the plane $x+2y+3z=4$ closest to the origin.

i.e. minimize $\underbrace{x^2+y^2+z^2}_{f(x,y,z)}$ subject to $x+2y+3z=4$

solve $\nabla f = \lambda \nabla g$ $\quad \nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla g = \langle 1, 2, 3 \rangle$

$$\left.\begin{array}{l} 2x=\lambda \\ 2y = 2\lambda \\ 2z = 3\lambda \end{array}\right\} \left.\begin{array}{l} x=\lambda/2 \\ y=\lambda \\ z=3\lambda/2 \end{array}\right\} \rightsquigarrow \frac{\lambda}{2}+2\lambda+\frac{9\lambda}{2}=4 \quad \lambda=\frac{4}{7}$$

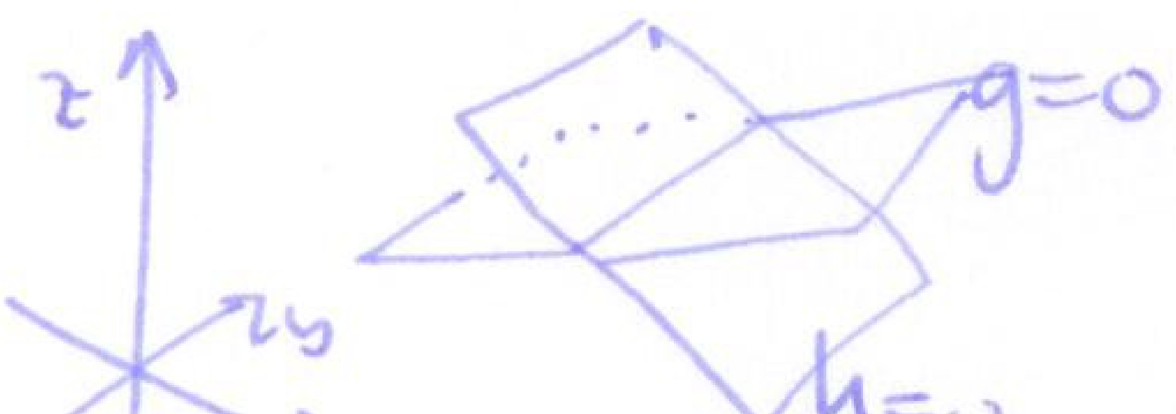
$x+2y+3z=4$

so point is $\langle \frac{2}{7}, \frac{4}{7}, \frac{6}{7} \rangle$

<u>Example</u> minimize $f(x,y,z)=x^2+y^2+z^2$ subject to $x+y=2$

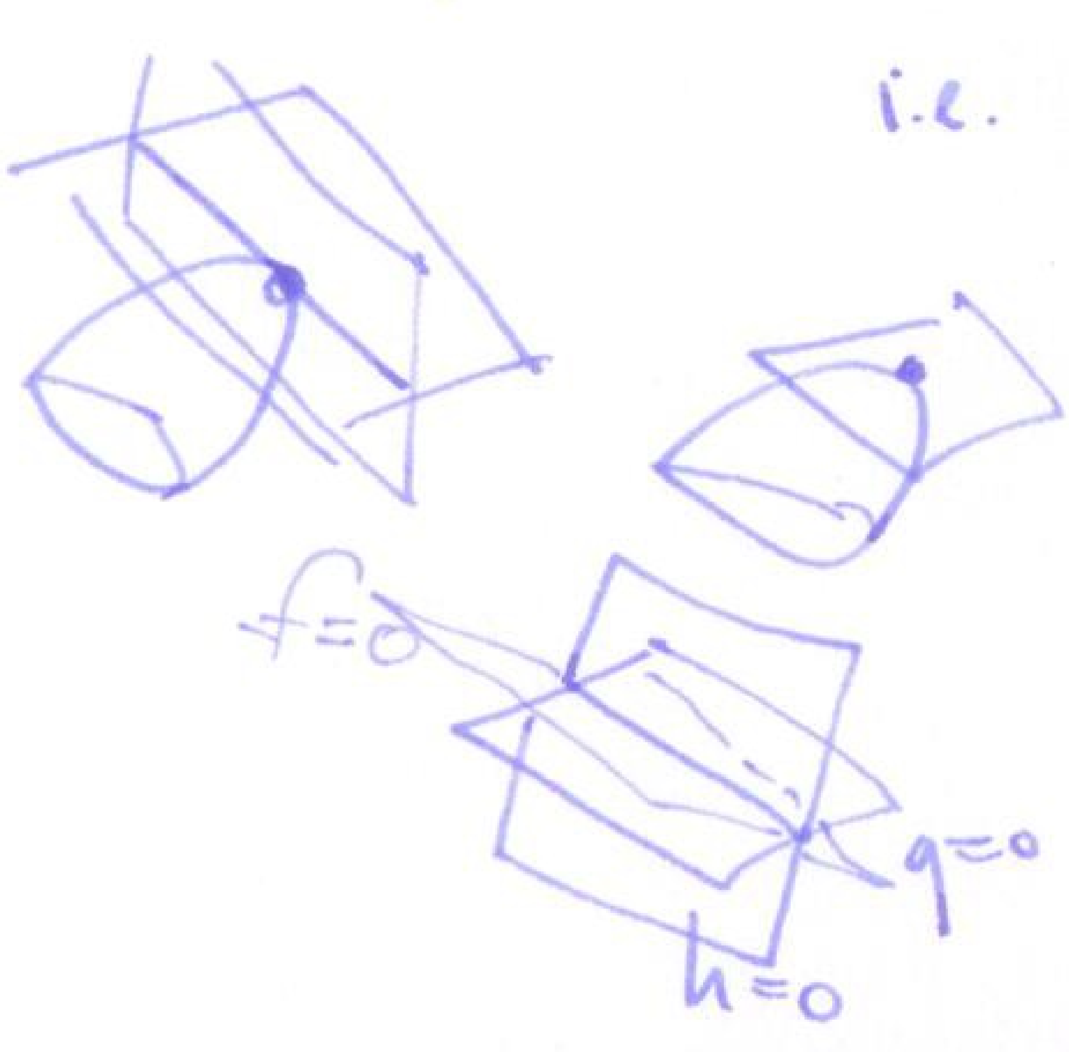
solve $\nabla f = \lambda \nabla g + \mu \nabla h$ $\quad$ <u>and</u> $y+z=4$.

why?



extreme points when level sets of f

tangent to $g=0 \cap h=0$



i.e. tangent direction to $g=0 \cap h=0 \subset$ tangent plane to $f=c$

i.e. normal vector to $f=c$ is a sum of normal vectors to $g=0$ and $h=0$.