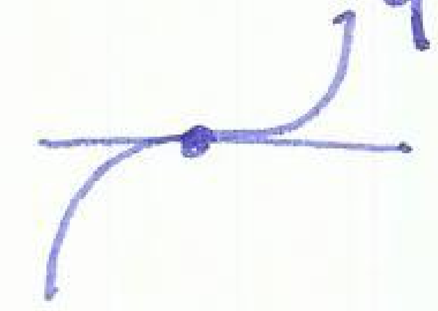
§14.7 Optimization

recall: 1 var

max, min $\Rightarrow \frac{dy}{dx} = 0$

$\frac{dy}{dx} = 0 \not\Rightarrow$ max, min



$y = x^3$.

2 vars local max  local min  $\Rightarrow$ flat tangent plane $z = \text{const}$.

recall: tangent plane to $z = f(x,y)$ at (a,b) is

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

flat tangent plane $\Rightarrow$ $f_x(a,b) = 0$ and $f_y(a,b) = 0$

warning $f_x(a,b)$ and $f_y(a,b) = 0 \not\Rightarrow$ local max or min.

Def A critical point (a,b) is a point such that $\frac{\partial f}{\partial x}(a,b) = 0$ and $\frac{\partial f}{\partial y}(a,b) = 0$, or at least one of $f_x(a,b)$, $f_y(a,b)$ does not exist.

Thm If $f(x,y)$ has a local max or min at (a,b), then $f_x(a,b) = 0$ and $f_y(a,b) = 0$.

Example finding critical pnts. $f(x,y) = x^2 - 2xy + 2y^2 + 3y + 1$

$$\left.\begin{array}{l} \frac{\partial f}{\partial x} = 2x - 2y \quad ① \\ \frac{\partial f}{\partial y} = -2x + 4y + 3 \quad ② \end{array}\right\}$$

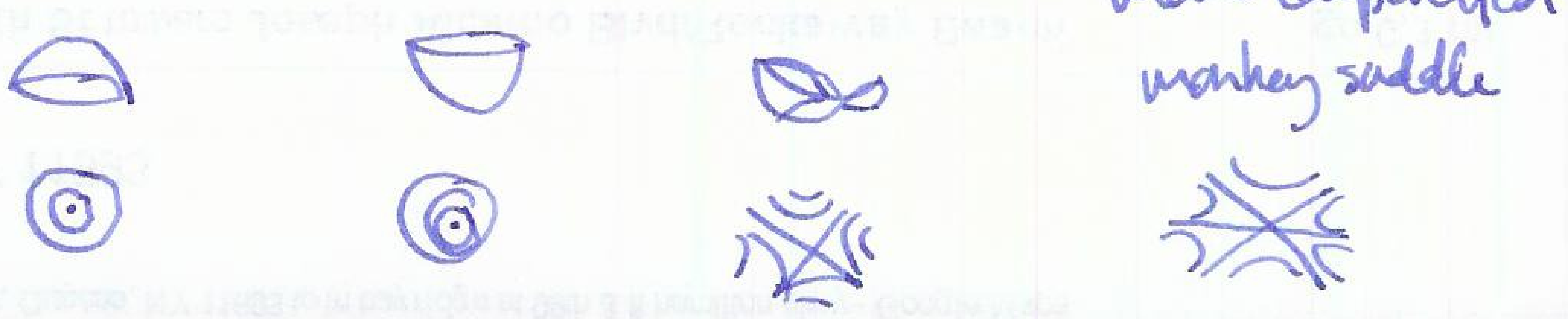
① + ②: $2y + 3 = 0$ $\quad y = -3/2 \quad x = -3/2$

only one critical point at $(-3/2, -3/2)$

Types of critical point: local max, local min, saddle, more complicated monkey saddle

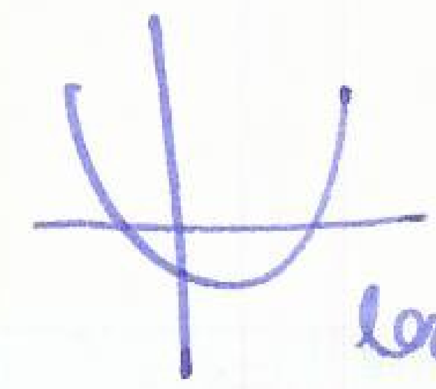
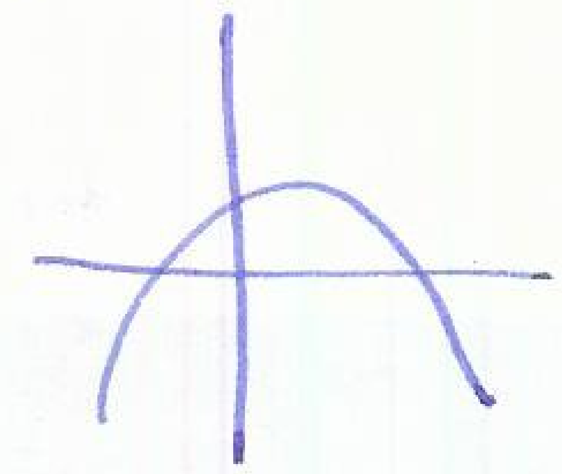
graphs

level sets



Q: How to tell which are? look at 2nd order / quadratic approx.

1 var: $f(x) \approx f(a) + \underbrace{f'(a)}_{=0 \text{ if critical point}}(x-a) + \frac{1}{2}f''(a)(x-a)^2 + O(x^3)$.

$f''(a) > 0$  local min $\qquad f''(a) < 0$  local max

$f''(a) = 0$ no information!

2 vars: Thm (2nd derivative test) Let (a,b) be a critical point for $f(x,y)$, and let $D(a,b) = f_{xx}(a,b)\,f_{yy}(a,b) - f_{xy}(a,b)^2$, then

1) if $D > 0$ then $f(a,b)$ is a <u>minimum</u> if $f_{xx}(a,b) > 0$
<u>maximum</u> if $f_{xx}(a,b) < 0$

2) if $D < 0$ then (a,b) is a saddle

3) if $D = 0$ no information.

Why does this work?

Quadratic approximation is $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \begin{bmatrix} x-a & y-b \end{bmatrix} \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} \begin{bmatrix} x-a \\ y-b \end{bmatrix}$

quadratic terms are: $f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) + f_{yy}(a,b)(y-b)^2$

this is a quadratic form like x^2+y^2 $\quad -(x^2+y^2)$ $\quad x^2-y^2$

up to change of coordinates this looks like

$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \leftrightarrow \lambda_1 x^2 + \lambda_2 y^2$

$\lambda_1, \lambda_2 > 0$	max
$\lambda_1, \lambda_2 < 0$	min
different sign	saddle
one $= 0$	no information.

Example $f(x,y) = (x^2+y^2)e^{-2x}$

$f(x,y) = (x^2+y^2)e^{-2x}$ $\quad f_x = 2xe^{-2x} + (x^2+y^2)\cdot -2e^{-2x}$

$= e^{-2x}(2x - 2x^2 - 2y^2).$

$f_y = 2ye^{-2x}$

find critical points: solve $\left.\begin{matrix} f_x = 0 \\ f_y = 0 \end{matrix}\right\} \Rightarrow y = 0 \Rightarrow 2x - 2x^2 = 0$ $\quad 2(x)(x-1) = 0$

$x = 0,\ x = 1.$ $\quad (0,0)$ $(1,0)$

$f_{xx} = -2e^{-2x}(2x - 2x^2 - 2y^2) + e^{-2x}(2-4x)$

$f_{xy} = -4ye^{-2x}$

$f_{yy} = 2e^{-2x}$

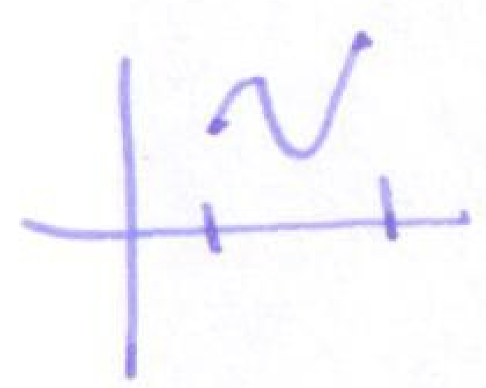
local min.

$D(0,0) = 2\cdot 2 - (-4\cdot 0) = 4$

$D(1,0) = 2\cdot 2 \ \ldots$

Absolute max/min

1d: a continuous function on a closed interval has an absolute max and min



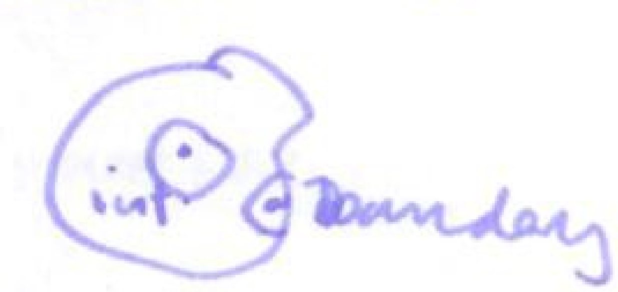
← abs max occurs at a max or endpoint
min min endpoint

2d: $D \subset \mathbb{R}^2$ $\quad f: \mathbb{R}^2 \to \mathbb{R}$

D is bounded if $D \subset$ disc of radius R about $(0,0)$

$x \in D$ is an interior point if there is a small disc around x contained in D

otherwise x is a boundary point.



interior of D = union of all interior points
boundary of D = union of all boundary points

D is closed if D contains all its boundary points

D is open if every point is an interior point.

Example

a b
$[a,b]$ closed

a b
open.
(a,b)

$\mathbb{R}$
closed and open.

$[a,b)$ neither closed nor open.

Thm $f(x,y)$ cts on $D \subset \mathbb{R}^2$ closed, bounded, then $f(x,y)$ has abs max/min on D
extreme values occur either at critical points in the interior, or on the boundary.