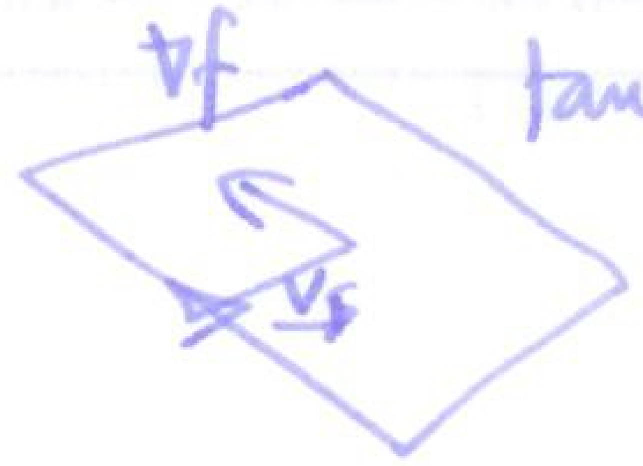
Thm The directional derivative $D_{\underline{v}} f$ is equal to

$$\boxed{D_{\underline{v}} f(a,b) = \nabla f(a,b) \cdot \underline{v}} \qquad (\|\underline{v}\| = 1!)\,.$$

sketch: ∇f tangent plane has equation



$$z = f(a,b) + \frac{\partial f}{\partial x}(a,b)(x-a) + \frac{\partial f}{\partial y}(a,b)(y-b)$$

so if $\underline{v} = \langle v_1, v_2 \rangle$ then rate of change is $\frac{\partial f}{\partial x}(a,b)v_1 + \frac{\partial f}{\partial x}(a,b)v_2$

$= \nabla f \cdot \underline{v}$

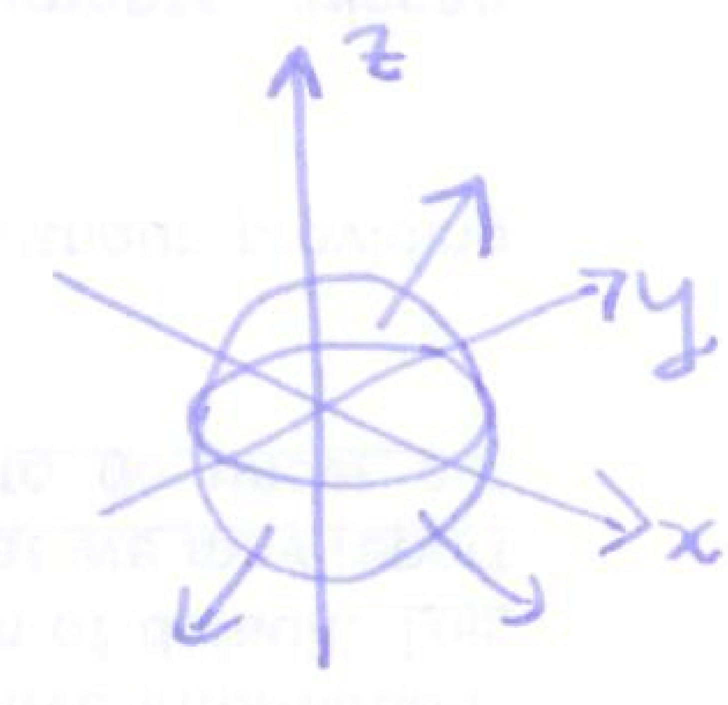
Useful properties

- if $\underline{v}$ is not a unit vector define $D_{\underline{v}} f(a,b) = \nabla f \cdot \underline{v}$
- $D_{\lambda \underline{v}} f(a,b) = \lambda \nabla f \cdot \underline{v}$

so if $\underline{v}$ is not a unit vector the directional derivative in direction $\underline{v}$

is $\frac{1}{\|\underline{v}\|} \nabla f \cdot \underline{v}$.

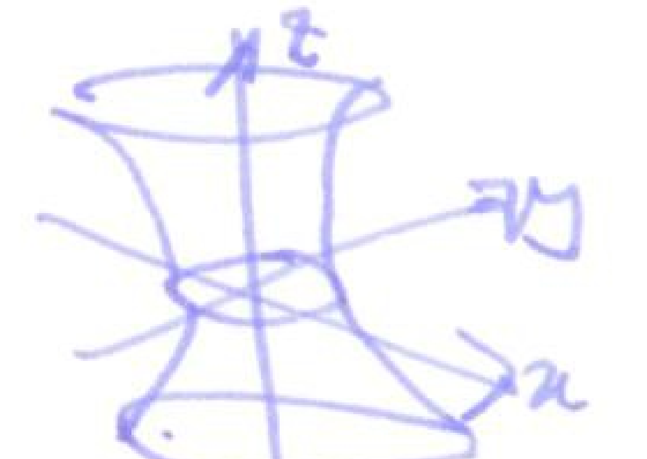
Applications



- Finding normal vectors: sphere $x^2+y^2+z^2=r^2$

consider $f(x,y,z) = x^2+y^2+z^2$

then $\nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$ is normal vector.



- Finding tangent planes: hyperboloid $x^2+y^2=z^2+1$.

find normal vectors: consider $f(x,y,z) = x^2+y^2-z^2 = r$

then $\nabla f = \langle 2x, 2y, -2z \rangle$ so normal vector at $(1,1,1)$ is $\langle 2, 2, -2 \rangle$

so tangent plane is $\underline{n} \cdot (\underline{x} - \underline{p}) = 0$ $\quad \langle 2,2,-2 \rangle \cdot (\langle x,y,z \rangle - \langle 1,1,1 \rangle) = 0$

$$2x + 2y - 2z = 2\,.$$

§14.6 Chain rule

recall $\mathbb{R} \xrightarrow{g} \mathbb{R} \xrightarrow{f} \mathbb{R}$

$x \longmapsto g(x) \longmapsto f(g(x))$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) = g'(x) \cdot f'(g(x))$$

general functions $f: \mathbb{R}^a \to \mathbb{R}^b$ $\quad f(x_1, x_2, \dots x_a) = (f_1(x_1, \dots, x_a), f_2(x_1, \dots x_a), \dots, f_b(x_1 \dots x_a))$

derivate $Df: \mathbb{R}^a \to \mathbb{R}^b$ is a linear map, given by the matrix of partial derivatives

$$Df = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_a} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_a} \\ \vdots & & & \\ \frac{\partial f_b}{\partial x_1} & \cdots & & \frac{\partial f_b}{\partial x_a} \end{bmatrix}}_{b \times a}$$

Examples $f: \mathbb{R}^3 \to \mathbb{R}$ $\quad f(x_1, x_2, x_3)$. $\quad Df = \left[\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3}\right] = \underline{\nabla} f$.

$f: \mathbb{R} \to \mathbb{R}^3$ $\quad x_1 \longmapsto (f_1(x_1), f_2(x_1), f_3(x_1))$ $\quad Df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} \\ \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_3} \end{bmatrix} = \underline{f}'(t)$.

Composition: $\mathbb{R}^a \xrightarrow{g} \mathbb{R}^b \xrightarrow{f} \mathbb{R}^c$

$\underline{x} \longmapsto \underline{f}(\underline{x}) \longmapsto g(f(x))$ $\quad f(g(x))$

$Dg \quad Dgf$

$$\underbrace{D(f(g(x)))}_{c \times a} = \underbrace{Df(g(x))}_{c \times b} \cdot \underbrace{Dg(x)}_{b \times a}$$

Example ① $\mathbb{R} \xrightarrow{g} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$

$x \mapsto (g_1(x), g_2(x), g_3(x)) \mapsto f(g_1(x), g_2(x), g_3(x))$.

$$\underbrace{D(f(g(x)))}_{1\times 1} = \underbrace{Df(g(x))}_{1\times 3} \underbrace{Dg(x)}_{3\times 1} = \underline{\nabla} f(g(x)) \cdot \underline{g}'(x) = \left\langle \frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right\rangle \cdot \langle g_1'(x), g_2'(x), g_3'(x) \rangle$$

②

$$\mathbb{R}^2 \xrightarrow{g} \mathbb{R}^2 \xrightarrow{f} \mathbb{R}^2$$
$$(x_1, x_2) \quad (y_1, y_2)$$

$$g(x_1, x_2) = (g_1(x_1, x_2), g_2(x_1, x_2))$$
$$f(y_1, y_2) = (f_1(y_1, y_2), f_2(y_1, y_2)).$$

$$D(f(g(x))) = Df(g(x)) \cdot Dg(x)$$

$$Df = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} \end{bmatrix} \qquad Dg = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} \\ \frac{\partial g_2}{\partial x_1} & \frac{\partial g_2}{\partial x_2} \end{bmatrix}$$

note: $Df \; Dg = \begin{bmatrix} \\ \end{bmatrix}\begin{bmatrix} \\ \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1}\frac{\partial g_1}{\partial x_1} + \frac{\partial f_1}{\partial y_2}\frac{\partial g_2}{\partial x_2} & \cdots \\ \cdots & \cdots \end{bmatrix}$

<u>mnemonic</u> $f(x_1, x_2, \ldots, x_n) \qquad x_i(y_1, y_2, \ldots y_m)$

to find $\frac{\partial f}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \frac{\partial f}{\partial x_2}\frac{\partial x_2}{\partial y_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_i}$

"differentiate f wrt to all variables, mult. by $\frac{\partial x_j}{\partial y_i}$ and add".

<u>Example</u> $f(x,y,z) = xy + z^3 \qquad x = s+t \qquad g: \mathbb{R}^2 \to \mathbb{R}^3$

$f: \mathbb{R}^3 \to \mathbb{R} \qquad y = s-t \qquad (s,t) \mapsto (x,y,z)$

$z = st$

so $f(g(s,t))$ makes sense.

<u>Q:</u> find $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$