observation: the gradient vector is perpendicular to the level sets / contours

2d:  3d: 

Useful properties

1) $\nabla(f+g) = \nabla f + \nabla g$

2) $\nabla(cf) = c\nabla f$ c constant.

3) product rule $\nabla(fg) = f\nabla g + g\nabla f$

4) chain rule $\nabla(F(f(x,y,z))) = F'(f(x,y,z))\nabla f$ $F(t)$ differentiable

check this makes sense $F\circ f$ $\mathbb{R}^3 \xrightarrow{f} \mathbb{R} \xrightarrow{F} \mathbb{R}$

$$\nabla(F(f(\underline{x}))) = \left\langle \frac{\partial}{\partial x}(F(f(\underline{x}))), \frac{\partial}{\partial y}(F(f(x,y,z))), \frac{\partial}{\partial z}(F(f(x,y,z))) \right\rangle$$

$$= \left\langle F'(f(\underline{x}))\cdot\frac{\partial f}{\partial x}, F'(f(\underline{x}))\cdot\frac{\partial f}{\partial y}, F'(f(\underline{x}))\cdot\frac{\partial f}{\partial y} \right\rangle$$

$$= F'(f(\underline{x}))\,\underline{\nabla f}.$$

Chain rule for paths

$f: \mathbb{R}^3 \to \mathbb{R}$ path $\underline{c}: \mathbb{R} \to \mathbb{R}^3$

$$t \mapsto (x(t), y(t), z(t))$$

composition: $\underline{c}\circ f$

$$\mathbb{R} \xrightarrow{\underline{c}} \mathbb{R}^3 \xrightarrow{f} \mathbb{R}$$

Thm $\frac{d}{dt}(f(\underline{c}(t))) = \nabla f(\underline{c}(t))\cdot \underline{c}'(t)$

in $\mathbb{R}^2$: $\frac{d}{dt}(f(\underline{c}(t))) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \langle x'(t), y'(t)\rangle = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$

<u>Example</u> temperature T varies with location like $T(x,y,z) = \dfrac{1}{x^2+y^2+z^2}$

if we move along an ellipse $\dfrac{x-3}{25} + \dfrac{y^2}{16} = 1$ (foci at $(0,0)$, $(6,0)$)

parameterize ellipse: $\underline{c}(t) = (5\cos t + 3,\ 4\sin t,\ 0)$

$$T(\underline{c}(t)) = \frac{1}{(5\cos t+3)^2 + (4\sin t)^2 + (0)^2}$$

$$\frac{d}{dt}\big(T(\underline{c}(t))\big) = \underline{\nabla} T \cdot \underline{c}'(t) = \left\langle \frac{\partial T}{\partial x}, \frac{\partial T}{\partial y}, \frac{\partial T}{\partial z} \right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \right\rangle$$

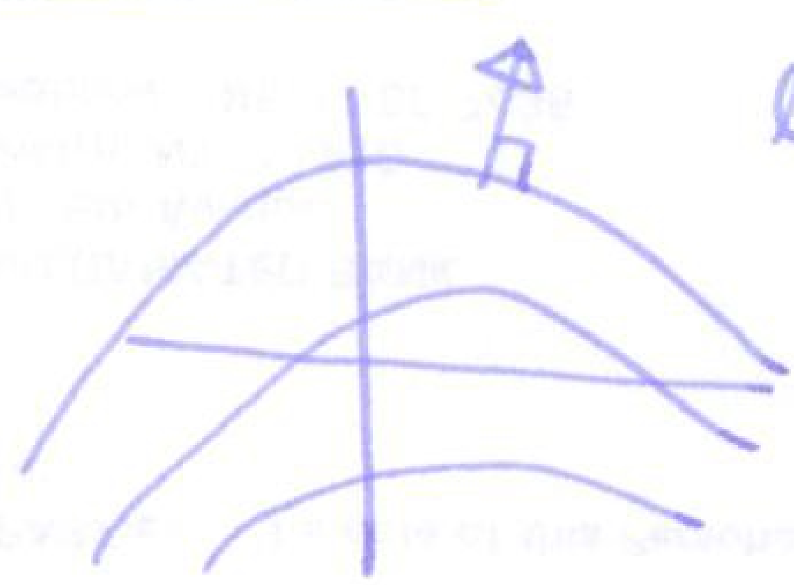
$$= \left\langle -(x^2+y^2+z^2)^{-2}\cdot 2x,\ -(x^2+y^2+z^2)^{-2}\cdot 2y,\ -(x^2+y^2+z^2)^{-2}\cdot 2z \right\rangle \cdot \langle -5\sin t, 4\cos t, 0\rangle$$

$$= \frac{-2}{(x^2+y^2+z^2)^2} \langle x, y, z\rangle \cdot \langle -5\sin t, 4\cos t, 0\rangle$$

$$= \frac{-2}{\left((5\cos t+3)^2 + (4\sin t)^2\right)^2} \langle 5\cos t+3, 4\sin t, 0\rangle \cdot \langle -5\sin t, 4\cos t, 0\rangle$$

Directional derivatives

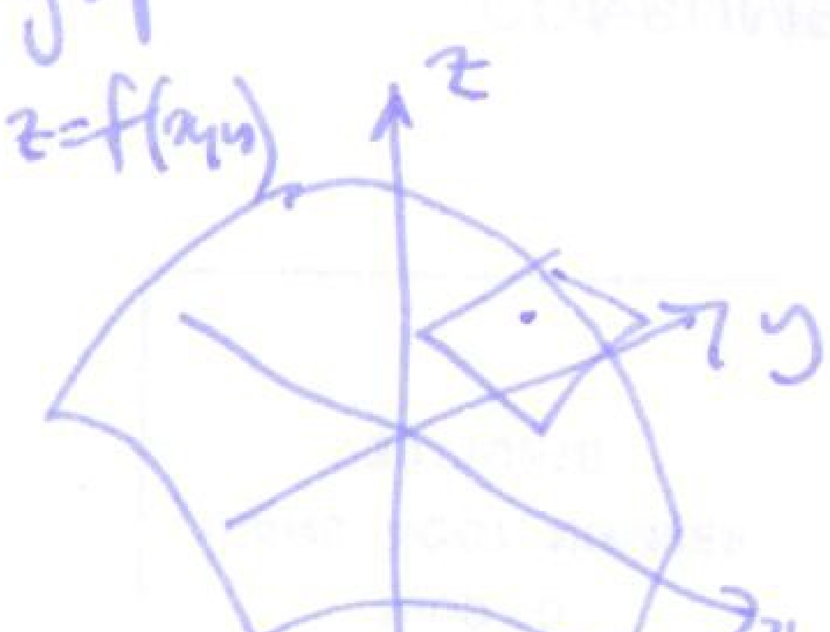
$f: \mathbb{R}^2 \to \mathbb{R}$



level sets $\quad \nabla f \perp$ level sets

- rate of change in direction $\nabla f = \|\underline{\nabla} f\|$.
- rate of change in $\perp$ direction to $\nabla f = 0$.

graph



tangent plane at a point

Q: what about rate of change in some other direction $\underline{v}$ (assume unit vector $\|\underline{v}\|=1$).