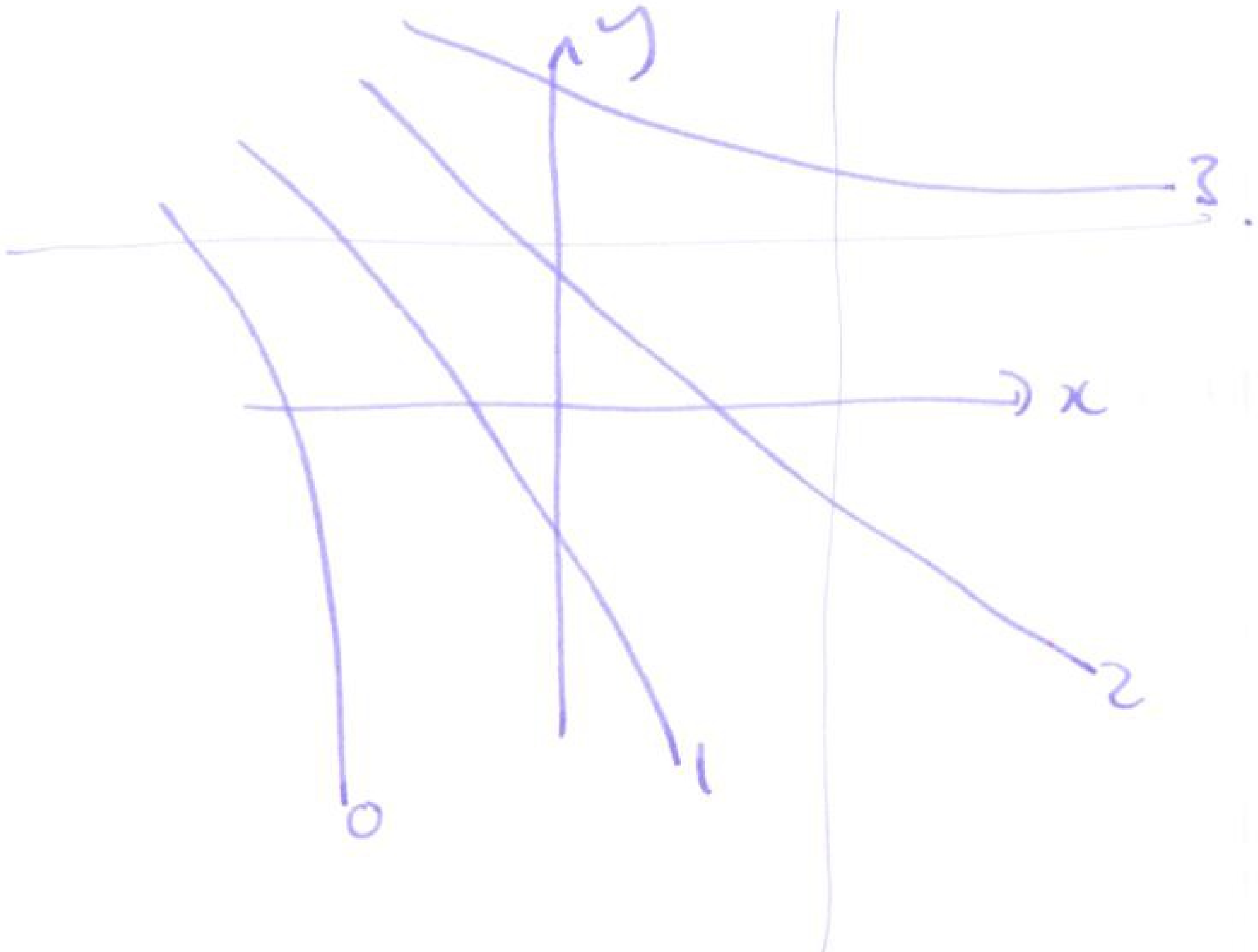
Interpreting contour maps / Level sets of $f(x,y)$.



$f_x > 0$ (numbers on contours increasing)

$f_{xx} < 0$ (contour lines getting further apart).

$f_y > 0$ (numbers on contour lines increasing)

$f_{yy} > 0$ (contour lines getting closer together)

$f_{xy} > 0$ (contours get closer together in x-direction as we move up in y-direction)

Examples ① $f(x,y) = x\sin(x+y)$

$f_x = \sin(x+y) + x\cos(x+y)$

$f_y = x\cos(x+y)$

② $f(x,y) = \dfrac{ae^{xy}}{y}$ $\quad f_x = \dfrac{aye^{xy}}{y}$ $\quad f_y = \dfrac{yaye^{xy} - ae^{xy}}{y^2}$.

Functions of 3 vars $f:\mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z)$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ "diff wrt z keeping x,y fixed".

Example $f(x,y) = xy + yz + xyz$

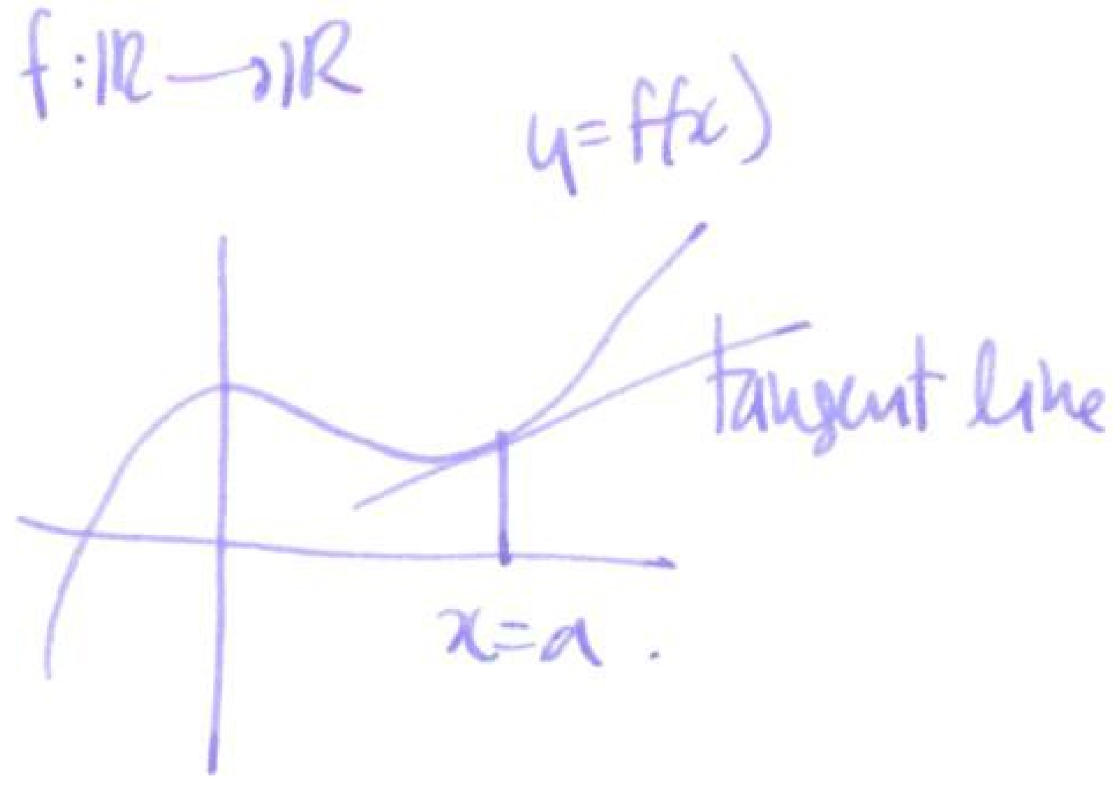
$f_x = y + yz$ $\quad f_y = x + z + xz$ $\quad f_z = y + xy$

Thm If 2nd order derivatives are cts then mixed partials are equal

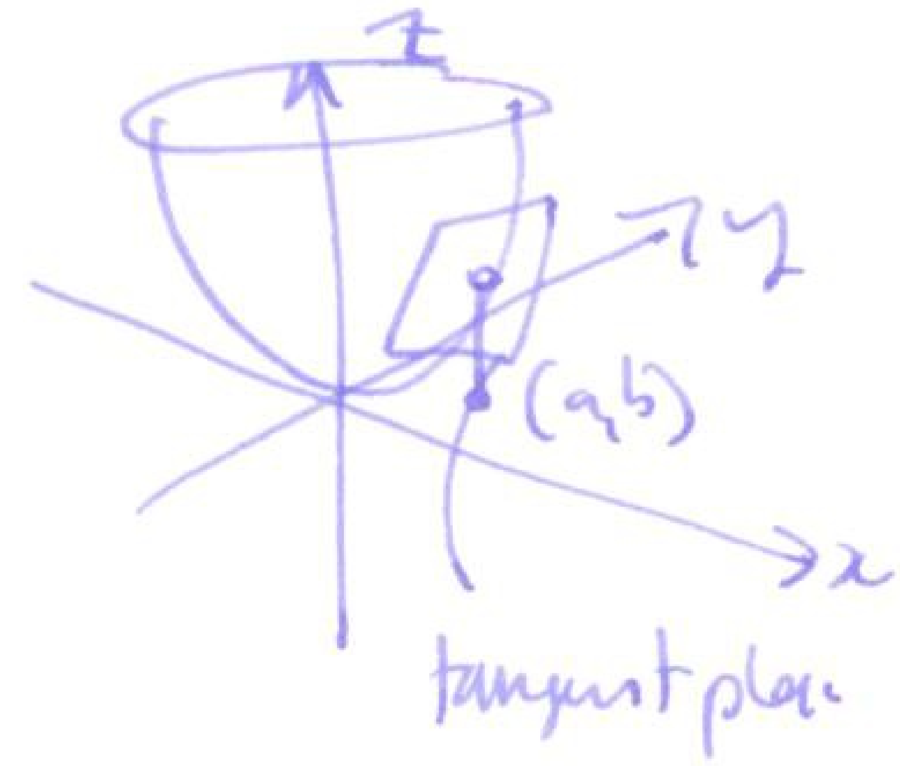
i.e. $f_{xy} = f_{yx}$ $\quad f_{xz} = f_{zx}$ etc.

§ 14.4 Differentiability, linear approximations and tangent planes

$f: \mathbb{R} \to \mathbb{R}$



$f: \mathbb{R}^2 \to \mathbb{R}$



linear approximation at $x=a$ is $L(x) = f(a) + f'(a)(x-a)$

tangent line is $y = L(x)$

linear approximation at $(x,y)=(a,b)$ is

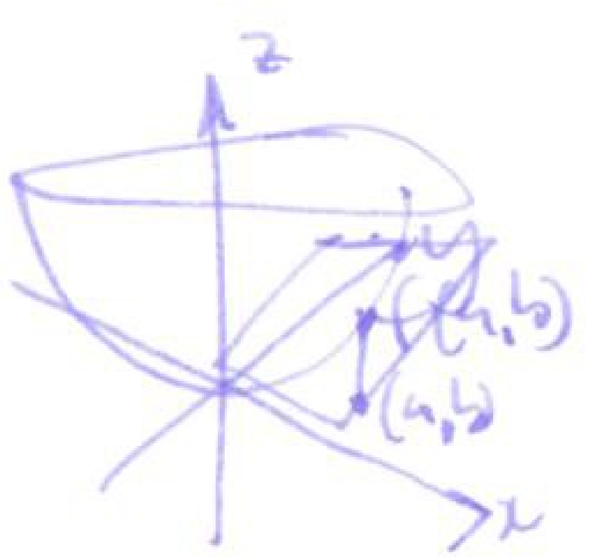
$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$$

tangent plane is $z = L(x,y)$.

<u>explanation</u> consider plane $z = cx + dy$

slope in x-direction is $\frac{\partial z}{\partial x} = c$

slope in y-direction is $\frac{\partial z}{\partial y} = d$



at (a,b) slope in x-direction is $\frac{\partial f}{\partial x}(a,b) = f_x(a,b)$

y-direction is $\frac{\partial f}{\partial y}(a,b) = f_y(a,b)$

Example find tangent plane to $f(x,y) = x^2+y^2$ at $(1,1)$

$f(1,1) = 2$ $\quad \frac{\partial f}{\partial x} = 2x$ $\quad \frac{\partial f}{\partial y} = 2y$ so $L(x,y) = f(1,1) + f_x(1,1)(x-1) + f_y(1,1)(y-1)$

$$L(x,y) = 2 + 2(x-1) + 2(y-1)$$

<u>Defn</u> $f(x,y)$ is <u>locally linear</u> at (a,b) if $L(x,y)$ approximates $f(x,y)$ to first order, i.e.

$$f(x,y) = L(x,y) + \underbrace{\epsilon(x,y)}_{\text{any function s.t. } \epsilon(x,y)\to 0 \text{ as } (x,y)\to(a,b)} \sqrt{(x-a)^2+(y-b)^2}$$

for (x,y) close to (a,b)

<u>Problem</u> f_x, f_y exist $\not\Rightarrow$ f locally linear.

Defn $f(x,y)$ is differentiable at (a,b) if

- $\frac{\partial f}{\partial x}(a,b)$ and $\frac{\partial f}{\partial y}(a,b)$ exist
- $f(x,y)$ is locally linear at (a,b)

in this case the tangent plane is $z = L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

Thm If $f_x(x,y)$ and $f_y(x,y)$ exist and are continuous close to (a,b) then $f(x,y)$ is differentiable at (a,b).

Bad example $z^2 = x^2+y^2$ not differentiable at $(0,0)$

$z = \sqrt{x^2+y^2}$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2x = \frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{r} = \cos\theta \\ \frac{\partial f}{\partial y} &= \frac{1}{2}(x^2+y^2)^{-1/2}\cdot 2y = \frac{y}{\sqrt{x^2+y^2}} = \frac{r\sin\theta}{r} = \sin\theta \end{aligned}\right\} \text{not cts at } (0,0)$$

§14.5 Gradient

2d: $f: \mathbb{R}^2 \to \mathbb{R}$ $f(x,y)$

the gradient of f is $\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$

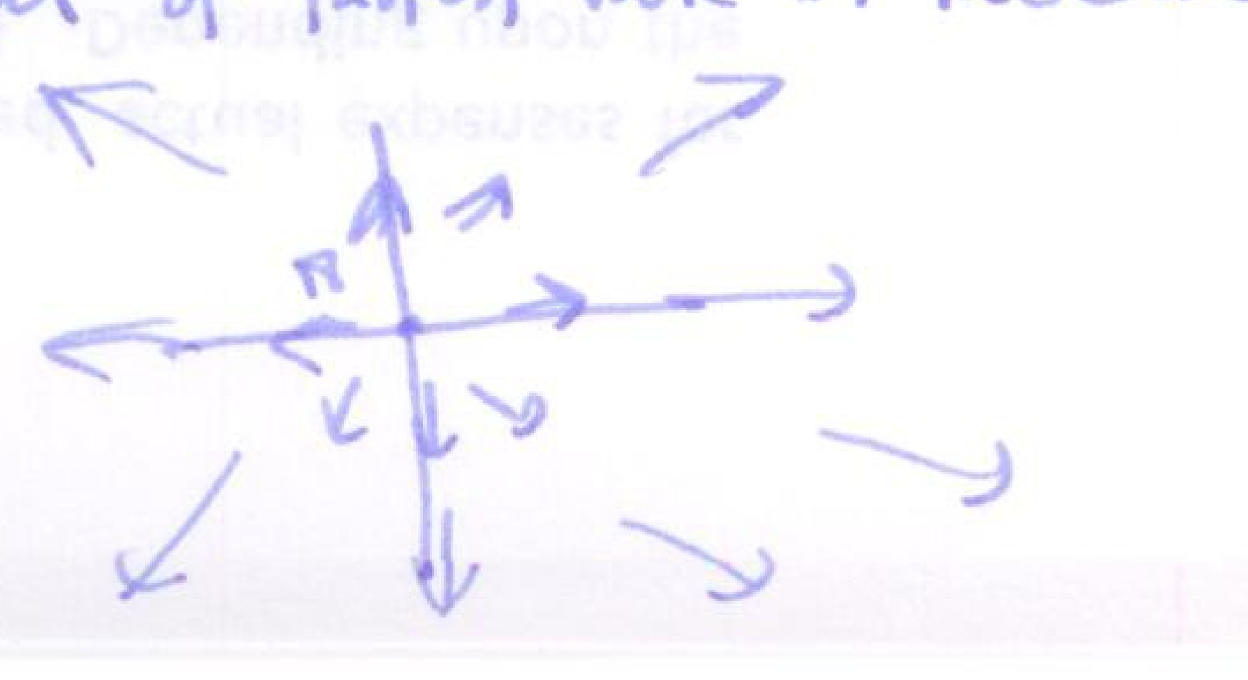
$$\nabla f(a,b) = \left\langle \frac{\partial f}{\partial x}(a,b), \frac{\partial f}{\partial y}(a,b) \right\rangle$$

3d: $f: \mathbb{R}^3 \to \mathbb{R}$ $f(x,y,z)$

$$\nabla f(a,b,c) = \left\langle \frac{\partial f}{\partial x}(a,b,c), \frac{\partial f}{\partial y}(a,b,c), \frac{\partial f}{\partial z}(a,b,c) \right\rangle$$

note: $\nabla f: \underset{\text{point}}{\mathbb{R}^2} \to \underset{\text{vector}}{\mathbb{R}^2}$ $\quad \nabla f: \underset{\text{point}}{\mathbb{R}^3} \to \underset{\text{vector.}}{\mathbb{R}^3}$

key facts:
- the gradient vector points in the direction of fastest rate of increase
- $\|\nabla f\|$ = fastest rate of increase



Example $f(x,y) = x^2+y^2$ $\quad \nabla f = \langle 2x, 2y \rangle$