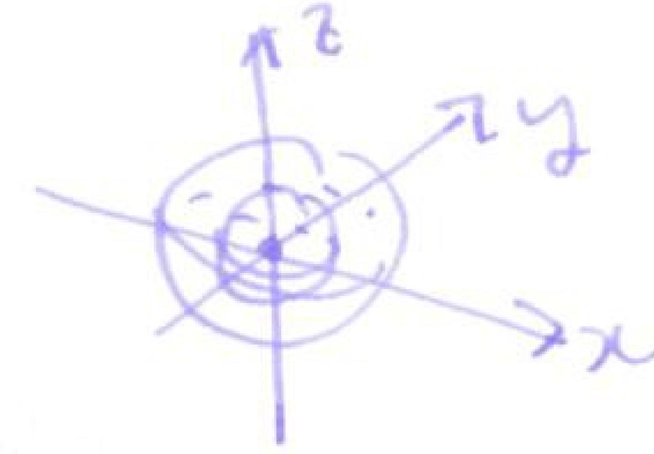
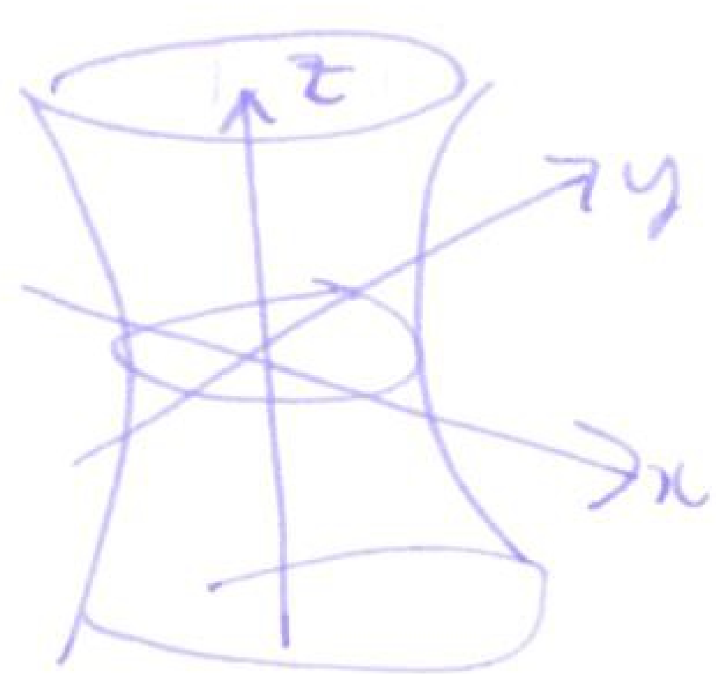
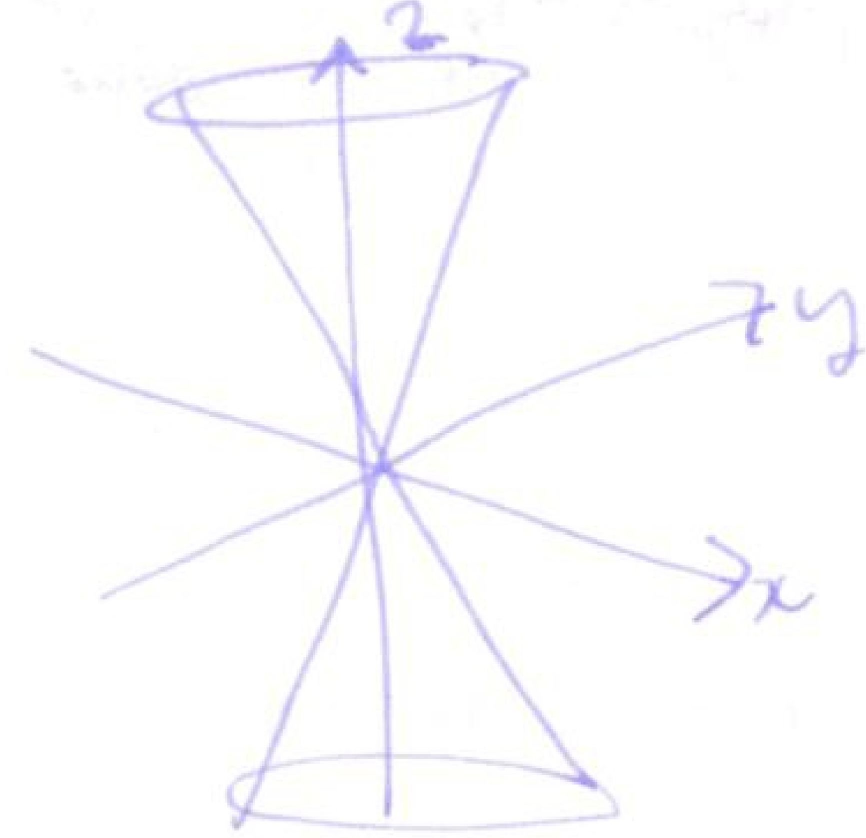
Example ① $f(x,y,z) = x^2+y^2+z^2$



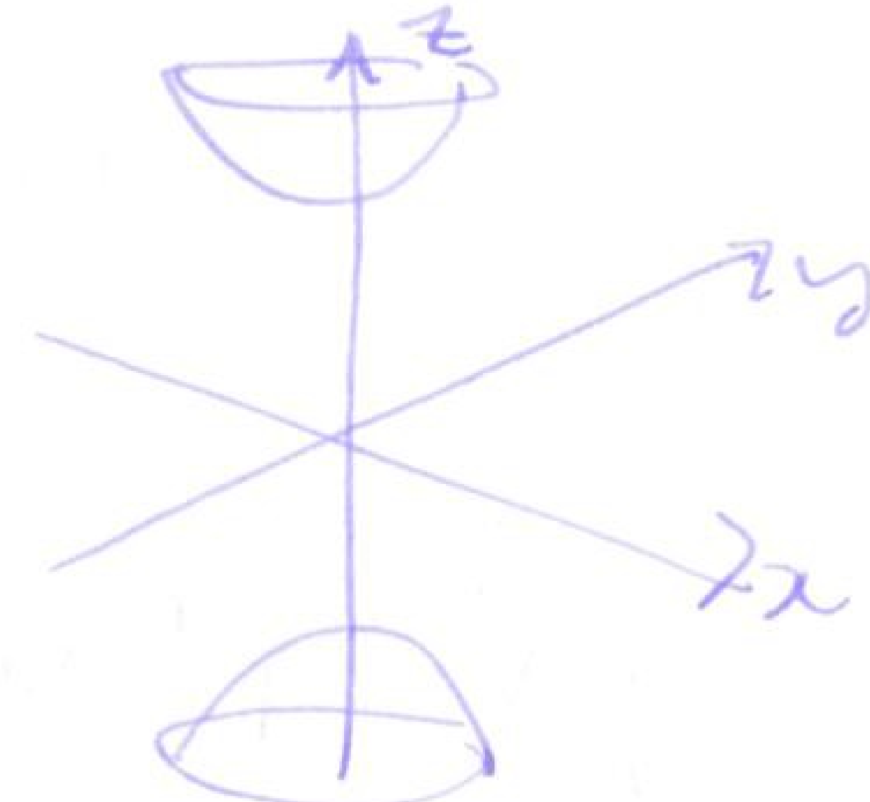
② $f(x,y,z) = x^2+y^2-z^2$



$f(x,y,z) = c > 0$



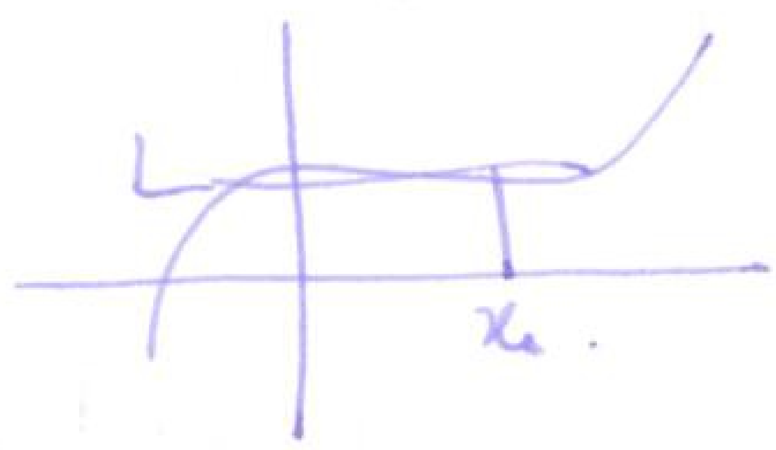
$f(x,y,z) = 0$



$f(x,y,z) = c < 0$

§14.2 Limits and continuity for functions of many variables

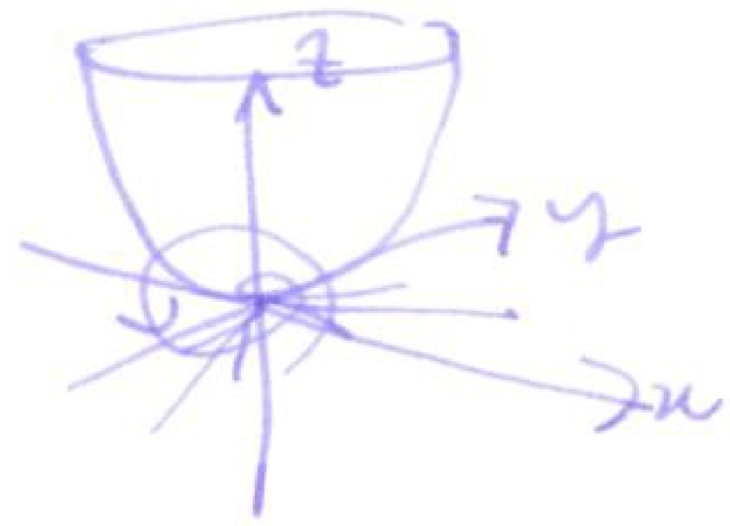
recall: $y = f(x)$



$\lim_{x \to x_0} f(x) = L$ if $|f(x) - L|$ get small as $|x_0 - x|$ get small

only two directions: left and right

2 vars $z = f(x,y)$



many ways to get to $(0,0)$

Defn $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$ if $|f(x,y) - L|$ gets small as $|(x,y)-(x_0,y_0)|$ gets small

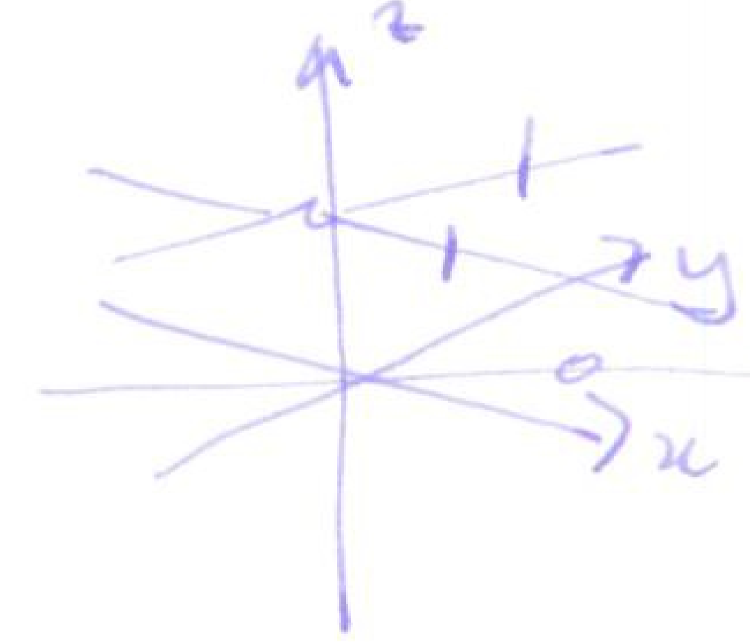
Bad example $f(x,y) = \left(\dfrac{x^2-y^2}{x^2+y^2}\right)^2$

Q what happens near $(0,0)$?

$$\lim_{(x,0)\to(0,0)} f(x,y) = \left(\frac{x^2}{x^2}\right)^2 = 1$$

$$\lim_{(y,0)\to(0,0)} f(x,y) = \left(\frac{-y^2}{y^2}\right)^2 = 1$$

$$\lim_{(x,x)\to(0,0)} f(x,y) = \left(\frac{0}{2x^2}\right) = 0$$



easier: use polars $x = r\cos\theta$, $y = r\sin\theta$

$$\left(\frac{r^2\cos^2\theta - r^2\sin^2\theta}{r^2\cos^2\theta + r^2\sin^2\theta}\right)^2 = \cos^2\theta - \sin^2\theta = \cos^2 2\theta.$$

Warning: $\lim_{(x,y)\to(0,0)} f(x,y) \not\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y)$ exists.

Thm Limit laws (assume $\lim_{(x,y)\to(a,b)} f(x,y)$ exists and $\lim_{(x,y)\to(a,b)} g(x,y)$ exists)

then:

sums: $$\lim_{(x,y)\to(a,b)} f(x,y)+g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) + \lim_{(x,y)\to(a,b)} g(x,y).$$

constant multiple: $$\lim_{(x,y)\to(a,b)} c f(x,y) = c \lim_{(x,y)\to(a,b)} f(x,y)$$

products: $$\lim_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = \lim_{(x,y)\to(a,b)} f(x,y) \lim_{(x,y)\to(a,b)} g(x,y)$$

quotients: $$\lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)}$$ as long as denominator $\neq 0$.

Defn continuity $f(x,y)$ is cts at (x_0,y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

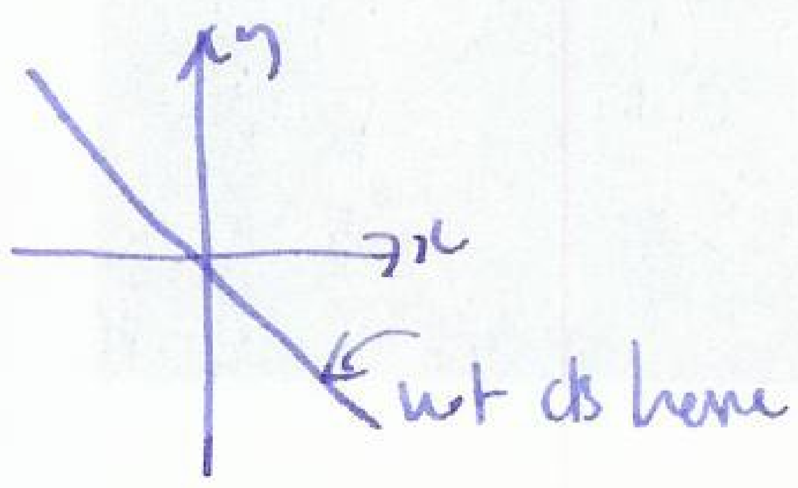
Q: when is $f(x,y)$ cts? A: hard to know.

however Thm "compositions of continuous functions are continuous".

so if f, g cts then fg cts
$f \pm g$ cts
f/g cts $(g \neq 0)$
$f \circ g$ cts.

<u>Example</u> $f(x,y) = \dfrac{x-y}{x+y}$ <u>note</u>: $f(x,y) = x$ cts, $f(x,y) = y$ cts $\Big\}$ linear functions are continuous.

so $x-y$ cts, $x+y$ cts $\quad \dfrac{x-y}{x+y}$ cts as long as $x+y=0$ i.e. $x \neq -y$.



How to show a limit does not exist: $f(x,y) = \dfrac{x^2}{x^2+y^2}$ at $(0,0)$.

just find some direction to get to $(0,0)$ which give different answers.

$$\left.\begin{aligned} \lim_{(x,0)\to(0,0)} \frac{x^2}{x^2+y^2} &= \lim_{(x,0)\to(0,0)} 1 = 1 \\ \lim_{(0,y)\to(0,0)} \frac{x^2}{x^2+y^2} &= \lim_{(0,y)\to(0,0)} 0 = 0 \end{aligned}\right\} \Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2+y^2} \text{ DNE}.$$

similar defns for functions of 3 or more vars.

<u>Thm</u> "compositions of cts functions are continuous"

so $f(x,y,z) = \dfrac{1}{x^2+y-z}$ cts where $x^2+y-z \neq 0$.

§14.3 Partial derivatives

$f:\mathbb{R}^2 \to \mathbb{R}$ $\qquad \dfrac{\partial f}{\partial x}(x,y) = f_x =$ "diff. wrt x keeping y constant"

$f(x,y)$ $\qquad \dfrac{\partial f}{\partial y}(x,y) = f_y =$ "diff wrt y keeping x constant".

Example $\quad f(x,y) = x^2+y^2 \qquad f_x = 2x \qquad f_y = 2y$

$f(x,y) = xy \qquad f_x = y \qquad f_x = x.$

$f(x,y) = xye^x + x^2y \qquad f_x = ye^x + xye^x + 2xy$

$f_y = xe^x + x^2$

note: f_x, f_y still functions of two variables, so we can partially differentiate again

$f_{xx} = \dfrac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x} \qquad f_{yx} = \dfrac{\partial^2 f}{\partial x \partial y} \qquad f_{yy} = \dfrac{\partial^2 f}{\partial y^2}.$

Example $\quad f_{xx} = ye^x + xye^x + ye^x + 2y$

$$\left.\begin{aligned} f_{xy} &= e^x + xe^x + 2x \\ f_{yx} &= xe^x + e^x + 2x \end{aligned}\right\} \text{equal!}$$

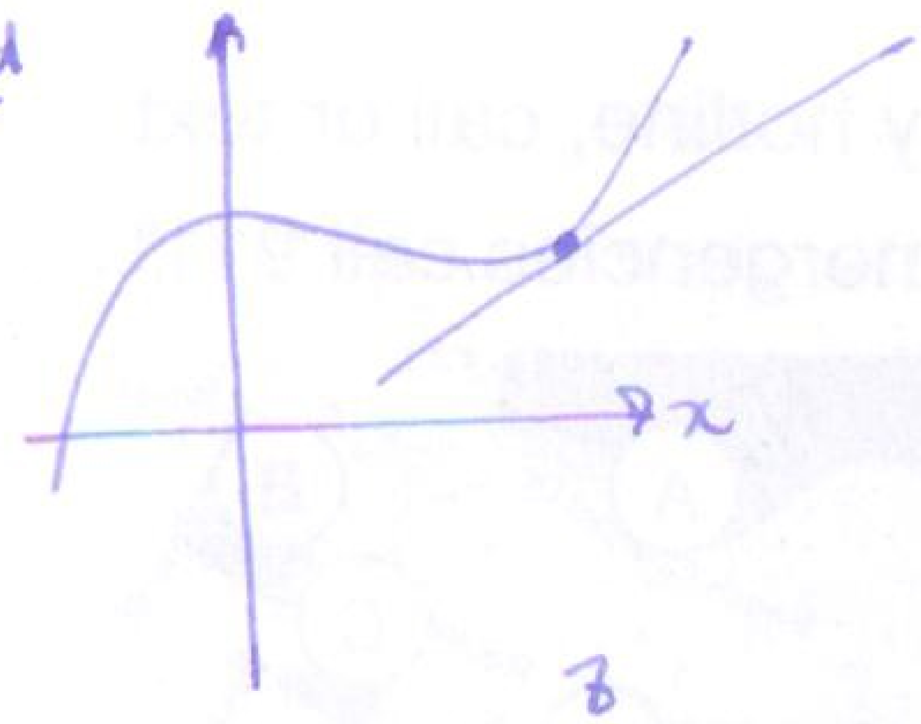
$f_{yy} = 0$

Thm If f_{xy} and f_{yx} are both continuous, then $f_{xy} = f_{yx}$

"mixed partials are equal"

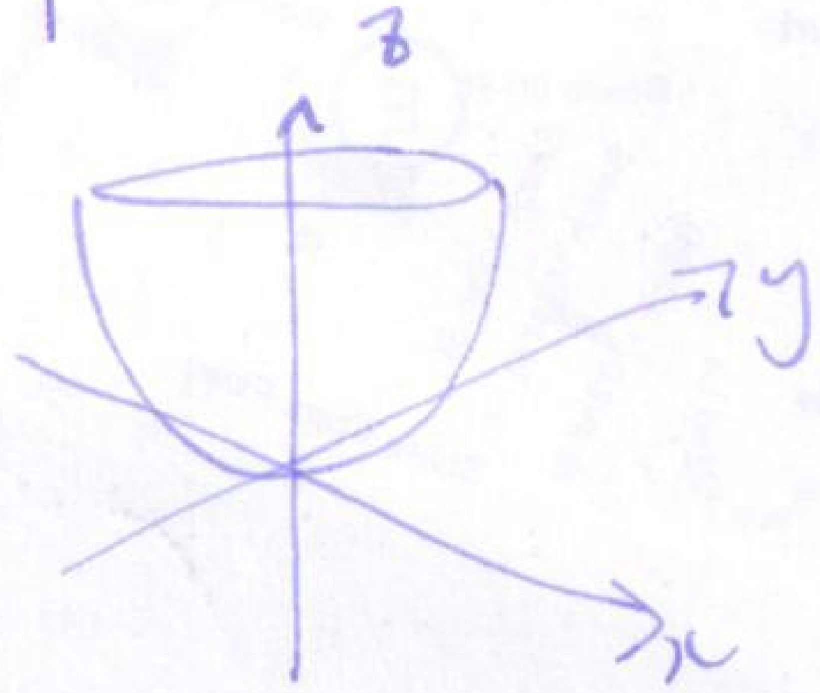
Q: what does this mean?

1d



$f:\mathbb{R}\to\mathbb{R}$ $\quad \frac{\partial f}{\partial x} = \frac{df}{dx}$ = rate of change | slope of tangent line.

2d



$f:\mathbb{R}^2\to\mathbb{R}$

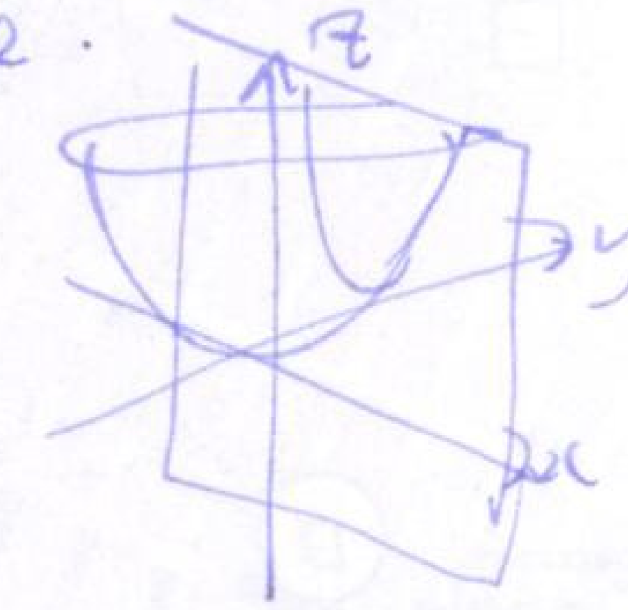
$f(x,y) = x^2+y^2$

$f_x(x,y) = 2x$

$f_y(x,y) = 2y$

f_x = diff^n wrt to x keeping y fixed i.e. $y=c$

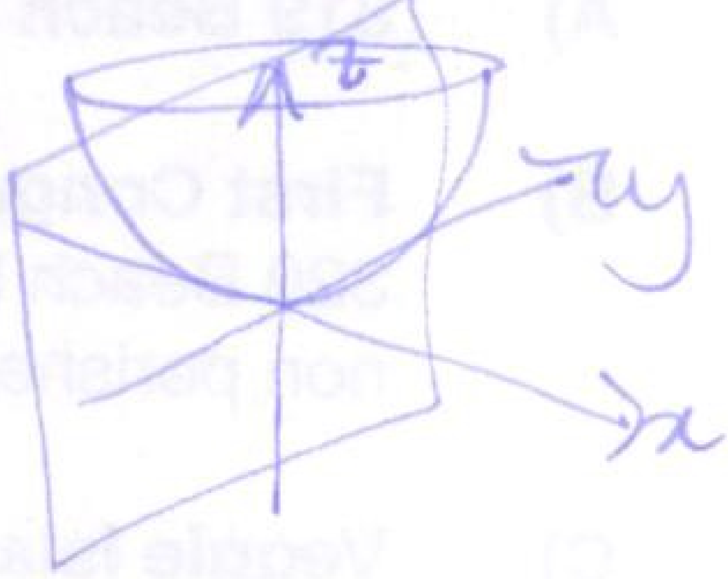
$\Leftrightarrow$ vertical slice parallel to xz-plane.



in $y=c$ $\quad f(x,c) = x^2+c^2$ (i.e. parabola with slope $2x$)

$\frac{\partial f}{\partial x}$ = slope in x-direction. $\quad \frac{\partial f}{\partial x}(x,y)$!

f_y = diff wrt to x keeping x fixed i.e. $x=c \Leftrightarrow$ vertical slice parallel to xz-plane



in $x=c$ $\quad f(c,y) = c^2+y^2$ (i.e. parabola with slope $2y$)

$\frac{\partial f}{\partial y}$ = slope in y-direction $\quad \frac{\partial f}{\partial y}(x,y)$!

f_{xx} = "2nd derivative in x-direction"

f_{yy} = "2nd derivative in y-direction"

f_{xy} = "rate of change of f_x in y-direction" $\Big\}$ equal if f_{xy} cts.

f_{yx} = "rate of change of f_y in x-direction"

Warning $\quad f_x, f_y$ exist $\not\Rightarrow$ f is continuous!