Observation we can go backwards, i.e. if we know acceleration $\underline{a}(t)$

then $\underline{v}(t) = \int_0^t \underline{a}(u)\,du + \underline{v}_0$ $\quad \underline{v}_0$ = initial velocity at $t=0$

$\underline{r}(t) = \int_0^t \underline{v}(t)\,du + \underline{r}_0$ $\quad \underline{r}_0$ = initial position at $t=0$

Example Find $\underline{r}(t)$ if $\underline{a}(t) = \langle 1, t\rangle$ $\quad \underline{v}_0 = \langle 1,1\rangle$ $\quad \underline{r}_0 = \langle 2,3\rangle$

$$\underline{v}(t) = \int_0^t \langle 1, u\rangle\,du + \underline{v}_0 = \left\langle \int_0^t du, \int_0^t u\,du \right\rangle + \langle 1,1\rangle$$

$$= \left\langle \left[u\right]_0^t, \left[\tfrac{1}{2}u^2\right]_0^t \right\rangle + \langle 1,1\rangle = \langle t, \tfrac{1}{2}t^2\rangle + \langle 1,1\rangle$$

$$= \langle t+1, \tfrac{1}{2}t^2+1\rangle$$

$$\underline{r}(t) = \int_0^t \langle u+1, \tfrac{1}{2}u^2+1\rangle\,du + \underline{r}_0$$

$$= \left\langle \left[\tfrac{1}{2}u^2+u\right]_0^t, \left[\tfrac{1}{6}u^3+u\right]_0^t \right\rangle + \langle 2,3\rangle$$

$$= \langle \tfrac{1}{2}t^2+t+2, \tfrac{1}{6}t^3+t+3\rangle$$

Newton's Laws of motion

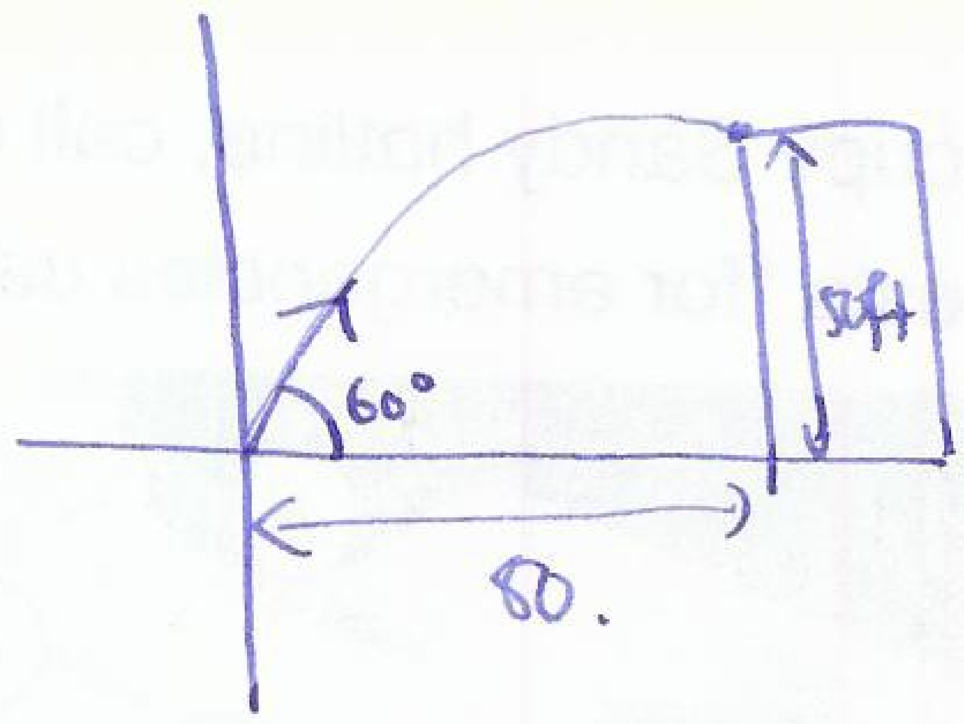
$F = ma$ $\quad$ vector form $\underline{F} = m\underline{a}$

note: if object has position $\underline{r}(t)$ this works if $\underline{F}$ not constant,

i.e. $\underline{F}$ can depend on position, $\underline{F}(\underline{x})$ or $\underline{F}(\underline{r}(t))$

then $\underline{F}(\underline{r}(t)) = m\,\underline{a}(t) = m\,\underline{r}''(t)$

Example




an object is thrown from the ground at an angle of 60°, what initial speed do you need to hit a point 50ft up 80 ft away?

gravity: downward force of magnitude $mg$ $\quad g = 32 ft/s$
$= 9.8 m/s \quad 10m/s.$

in vector form $\underline{F} = \langle 0, -gm \rangle$

$\underline{F} = m\underline{r}''(t) \Rightarrow \quad \underline{r}''(t) = \langle 0, -g \rangle$

integrate wrt $t$: $\quad \underline{r}'(t) = \int_0^t \underline{r}''(u)\,du = \int_0^t \langle 0, -g \rangle\,du$

$$= \langle 0, -32t \rangle + \underline{v_0}$$

integrate wrt $t$: $\quad \underline{r}(t) = \int_0^t \underline{r}'(u)\,du = \int_0^t \langle 0, -32u \rangle + \underline{v_0}\,du.$

$$= \int_0^t \left[ \langle 0, -16u^2 \rangle + \underline{v_0}u \right]_0^t + \underline{r_0}.$$

$$= \langle 0, -16t^2 \rangle + \underline{v_0}t + \underline{r_0}.$$

$\underline{r_0} = \underline{0} \qquad \underline{v_0} = v_0 \langle \cos 60°, \sin 60° \rangle = v_0 \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

so $\quad \underline{r}(t) = \langle 80, 50 \rangle = \langle 0, -16t^2 \rangle + tv_0 \langle \frac{1}{2}, \frac{\sqrt{3}}{2} \rangle$

$80 = \frac{1}{2} t v_0 \Rightarrow \quad t = \frac{160}{v_0}$

$50 = -16t^2 + \frac{\sqrt{3}}{2} t v_0 \Rightarrow \qquad 50 = -16 \cdot \left(\frac{160}{v_0}\right)^2 + \frac{\sqrt{3}}{2} \frac{160}{v_0} v_0.$

$$v_0^2 = \frac{16 \cdot (160)^2}{80\sqrt{3} - 50} \approx .$$

§14.1 functions of many variables

**Examples** height above sea level $h(x,y)$
temperature $t(x,y,z)$

**Def^n** A function of many variables is a function $f: \mathbb{R}^n \to \mathbb{R}$ (or domain $D \subset \mathbb{R}^n$)
domain range.

**Examples**

$$f: \mathbb{R}^2 \to \mathbb{R}$$
$$(x,y) \mapsto \sqrt{9-x^2-y^2}$$

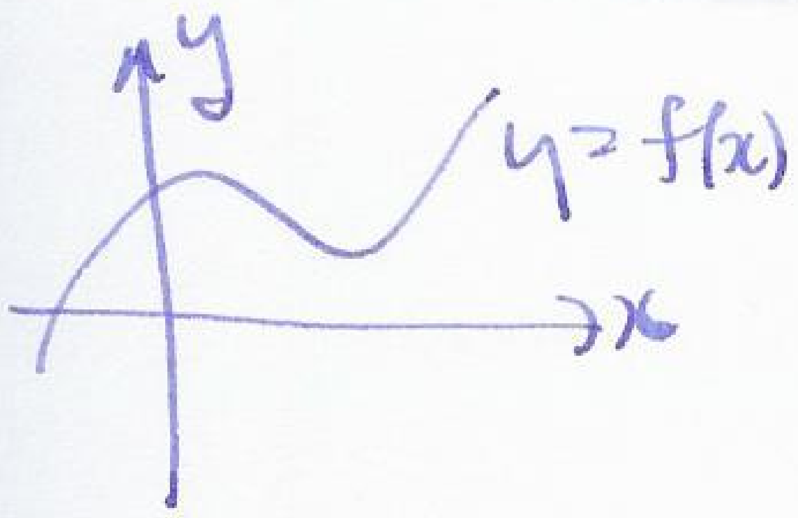
Q: what is $D$?

$$f: \mathbb{R}^3 \to \mathbb{R}$$
$$(x,y,z) \mapsto x\sqrt{y} + \ln(z-1)$$

Q: what is $D$?

**Recall** a function of one variable has a graph.

$f: \mathbb{R} \to \mathbb{R}$
$x \mapsto f(x)$.

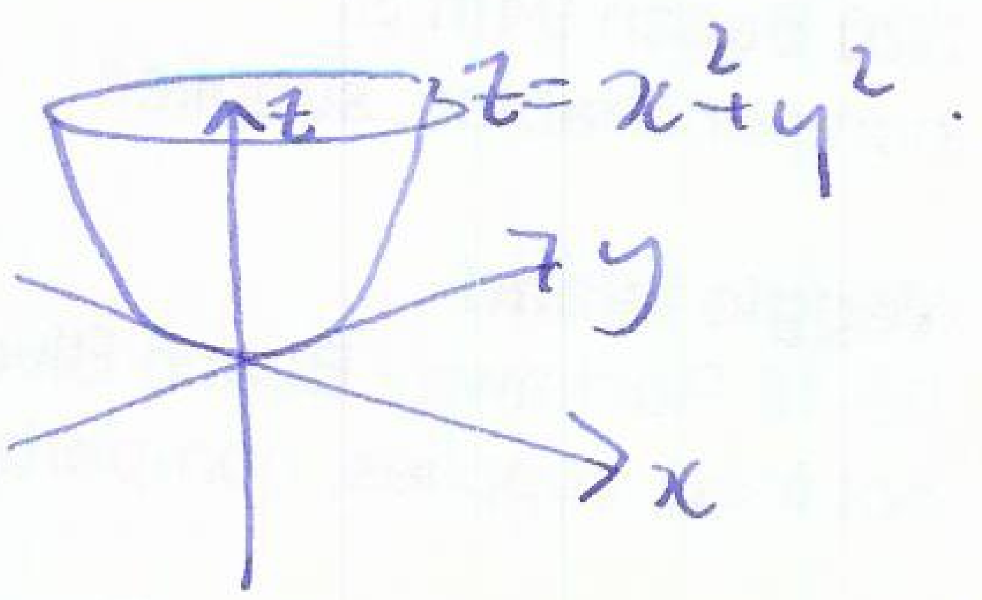



graph is $(x, f(x))$ in $\mathbb{R}^2$

Similarly we can draw the graph of a function ~~of~~ two variables.

$f: \mathbb{R}^2 \to \mathbb{R}$
$(x,y) \mapsto f(x,y)$

graph is $(x,y,f(x,y))$
i.e. a surface in $\mathbb{R}^3$.




**Example** ① draw graph of $f(x,y) = x^2+y^2$

traces: horizontal: solve $f(x,y) = x^2+y^2 = c$ (circles of radius $\sqrt{c}$)
vertical: fix $x=c$ or $y=c$.
$f(x,c) = x^2+c^2$ (parabola)
$f(c,y) = c^2+y^2$ (parabola)

② draw graph of $f(x,y) = x^2-y^2$

vertical traces are: $f(x,c) = x^2-c^2$ ∪
$f(c,y) = c^2-y^2$ ∩.

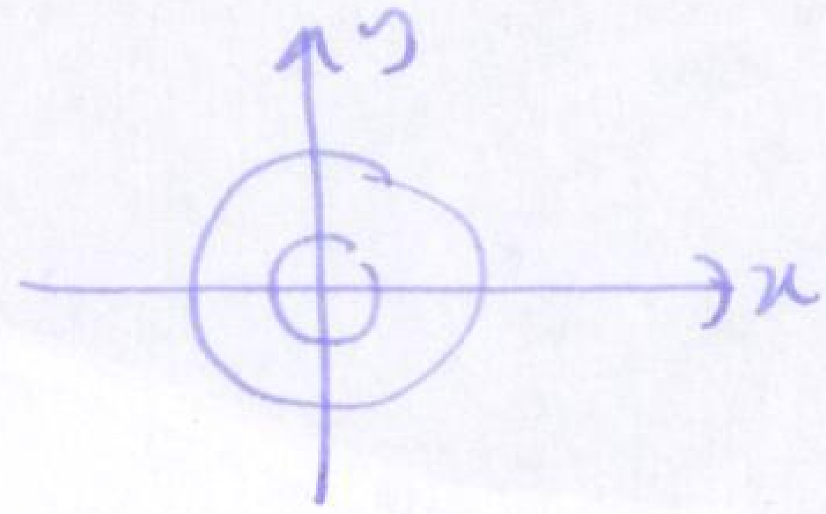
graph is plane $z = ax+by+c$.
$f(x,y) = ax+by+c$ ... traces are straight lines.

## Contour maps

The horizontal traces are known as <u>contour lines</u> or <u>level sets</u>.

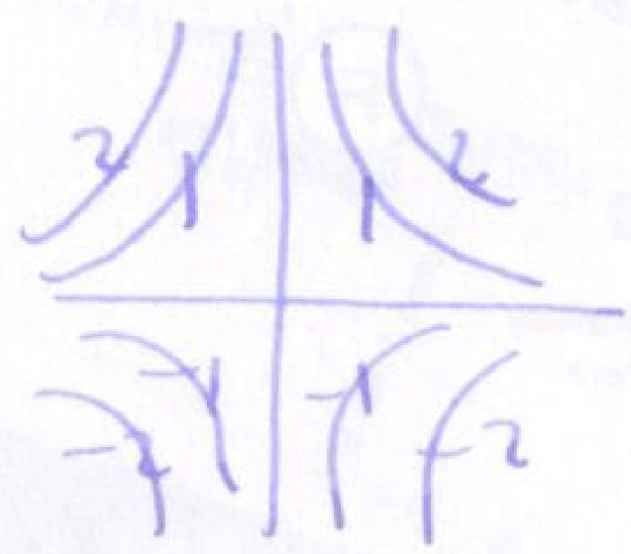
($\Leftrightarrow$ solutions to $f(x,y)=c$.) contour lines / level sets live in the <u>domain</u>!

<u>Example</u> $f(x,y)=\sqrt{4-x^2-y^2}$




② $f(x,y)=x^2y$

$f(x,y)=c \Leftrightarrow x^2y=c \quad y=\frac{c}{x^2}$




## On the interpretation of contours


far apart
$\Updownarrow$
slow change.


close together
$\Updownarrow$
fast change.

direction of fastest change is $\perp$ to contour.

$\parallel$ to contour function stays the same.

special patterns:

local max/min depending on labels.


saddle

average rate of change $\frac{\Delta f}{\Delta \underline{r}}$


## Functions of 3 vars

graphs: live in $\mathbb{R}^4$, hard to draw.

level sets: $f(x,y,z)=c$ are surfaces in $\mathbb{R}^3$, so we can draw these