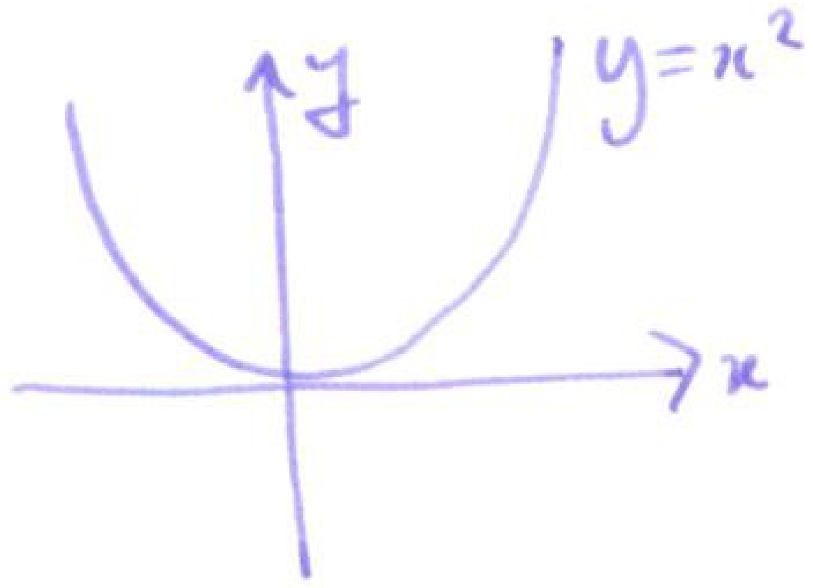
Example



$\underline{r}(t) = \langle t, t^2 \rangle$

$\underline{r}'(t) = \langle 1, 2t \rangle$

note $\|\underline{r}'(t)\| \neq 0$ in this example. Q $\langle t^3, t^6 \rangle$?

Integration

Def^n We define integration to be componentwise, i.e. $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$

$$\int_a^b \underline{r}(t)\,dt = \left\langle \int_a^b x(t)\,dt, \int_a^b y(t)\,dt, \int_a^b z(t)\,dt \right\rangle$$

the integral exists if each of the components is integrable.

Example $\displaystyle\int_0^1 \left\langle t, e^{2t}, \frac{1}{1+x^2} \right\rangle dx = \left\langle \int_0^1 t\,dt, \int_0^1 e^{2t}\,dt, \int_0^1 \frac{1}{1+x^2}\,dx \right\rangle.$

$$= \left\langle \left[\tfrac{1}{2}t^2\right]_0^1, \left[\tfrac{1}{2}e^{2t}\right]_0^1, \left[\tan^{-1}(x)\right]_0^1 \right\rangle = \left\langle \tfrac{1}{2}, \tfrac{1}{2}e^2 - \tfrac{1}{2}, \underbrace{\tan^{-1}(1)}_{\pi/4} \right\rangle.$$

Antiderivatives

Def^n An anti-derivative of $\underline{r}(t)$ is a function $\underline{R}(t)$ s.t. $\underline{R}'(t) = \underline{r}(t)$

Th^m 4 Let $\underline{R}_1(t)$ and $\underline{R}_2(t)$ be antiderivatives of $\underline{r}(t)$ (i.e. $\underline{R}_1'(t) = \underline{R}_2'(t) = \underline{r}(t)$). Then $\underline{R}_1(t) = \underline{R}_2(t) + \underline{c}$ $\underline{c}$ constant vector.

General antiderivative $\displaystyle\int \underline{r}(t)\,dt = \underline{R}(t) + \underline{c}$

Fundamental theorem of calculus (vector valued version)

$\underline{r}(t)$ continuous on $[a,b]$ and $\underline{R}(t)$ an antiderivative of $\underline{r}(t)$.

Then $\displaystyle\int_a^b \underline{r}(t)\,dt = \underline{R}(b) - \underline{R}(a).$

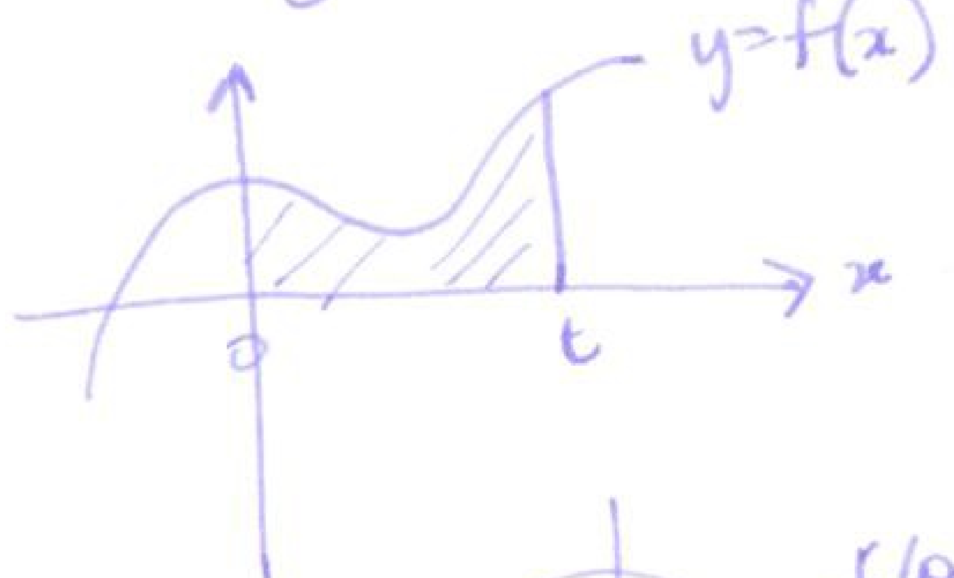
Example A particle moves with velocity $\underline{r}(t) = \langle t, \sin t \rangle$ if it starts at $\langle 1,1 \rangle$ at $t=0$, where is it at $t=4$?

$$\int_0^4 \underline{r}(t)\,dt = \underline{R}(4) - \underline{R}(0) \qquad \underline{R}(t) = \langle \tfrac{1}{2}t^2, -\cos t \rangle$$

$$= \langle 8, -\cos 4 \rangle - \langle 0, -1 \rangle = \langle 8, 1-\cos 4 \rangle.$$

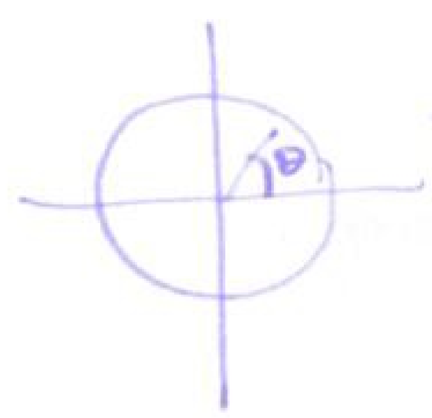
$\underline{r}$ position $\downarrow \frac{d}{dt}$ $\uparrow \int$.
$\underline{r}'$ velocity $\downarrow \frac{d}{dt}$ $\uparrow \int$
$\underline{r}''$ acceleration / $\underline{F} = m\underline{a}$

Warning / motivation $f: \mathbb{R} \to \mathbb{R}$

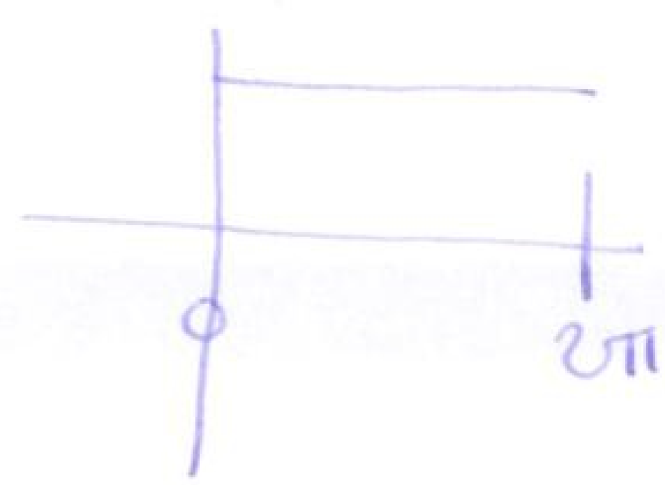


has an integral $F(x) = \int_0^x f(t)\,dt$ = area under the curve

note this doesn't work for f: circle $\to \mathbb{R}$.



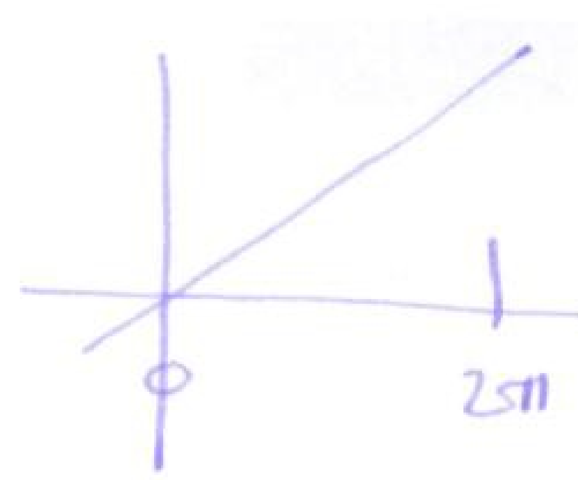
$f(\theta)$, $f(0) = f(2\pi)$



e.g. $f(\theta) = 1$.

$f'(\theta) = 0$.

$F(t) \int_0^t f(\theta)\,d\theta = t$?



not periodic! $F(0) = 0$
$F(2\pi) = 2\pi$ ※.

§13.3 Arc length and speed

Recall: arc length of curves in $\mathbb{R}^2$

$(x(t), y(t))$

paramaterized curve $(x(t), y(t))$ $t \in [a,b]$.

length $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt$

this generalizes to parameterized curves in $\mathbb{R}^3$: $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ $t \in [a,b]$

length $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\,dt = \int_a^b \|\underline{r}'(t)\|\,dt$

Example Find the arc length of $\underline{r}(t) = \langle \cos 2t, \sin 2t, 2t \rangle$ for $t \in [0, 2\pi]$

$\underline{r}'(t) = \langle -2\sin 2t, +2\cos 2t, 2 \rangle$.

$\|\underline{r}'(t)\| = \sqrt{4\sin^2 2t + 2\cos^2 2t + 4} = \sqrt{4+4} = 2\sqrt{2}$

$$L = \int_0^{2\pi} 2\sqrt{2}\,dt = 2\sqrt{2}\,[t]_0^{2\pi} = 4\pi\sqrt{2}.$$

<u>Observation</u> $\underline{r}'(t)$ is tangent to the curve (as long as $\underline{r}'(t) \neq 0$)

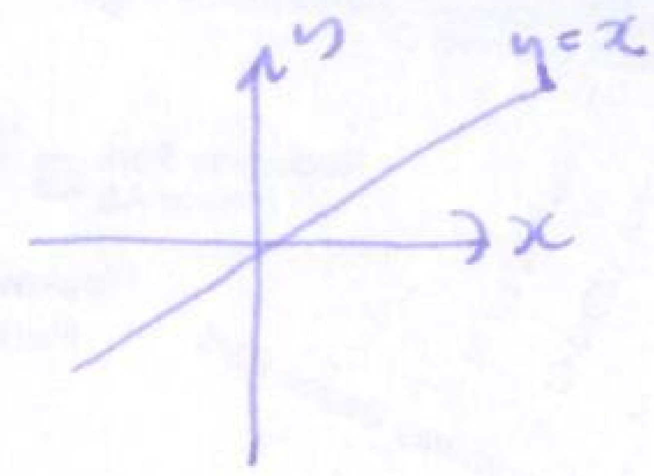
$\|\underline{r}'(t)\|$ is the <u>speed</u> at time t.

<u>Example</u> If a particle moves with position given by $\underline{r}(t) = \langle e^{2t}, t^{1/3}, \tan t \rangle$ find the speed at $t=2$: $\underline{r}'(t) = \langle 2e^{2t}, \frac{1}{3}t^{-2/3}, \sec^2 t \rangle$

$$\|\underline{r}'(2)\| = \cancel{\sqrt{4e}}\ \left\| \left\langle 2e^4, \frac{1}{3 \cdot 2^{2/3}}, \sec^2(2) \right\rangle \right\| = \sqrt{4e^8 + \frac{1}{9 \cdot 2^{4/3}} + \sec^4(2)}.$$

<u>Arc length parameterizations</u>

<u>Problem</u>: parameterizations are not unique.



$\underline{r}(t) = \langle t, t \rangle$

$\underline{r}(t) = \langle t^3, t^3 \rangle$.

<u>Special parameterizations</u>: <u>arc length</u> or <u>unit speed</u> parameterizations.

<u>Def^{n}</u> $\underline{r}(t)$ is an arc length parameterization if $\|\underline{r}'(t)\| = 1$ for all t.
(i.e. you move along the curve with unit speed).

<u>Example</u> find an arc length parameterization for $\underline{r}(t) = \langle 2t, 1-2t, t \rangle$

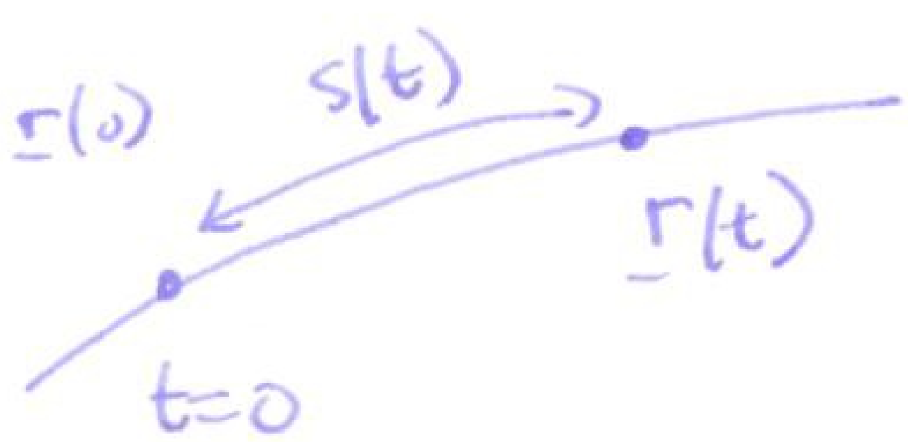
① find arc length at time t:

$$s(t) = \int_0^t \|r'(u)\|\,du \qquad \underline{r}'(t) = \langle 2, -2, 1 \rangle$$

$$\|\underline{r}'(t)\| = \sqrt{4+4+1} = 3.$$

so $$s(t) = \int_0^t 3\,du = \left[3u\right]_0^t = 3t$$

② find the inverse function for arc length



instead of $\underline{r}(t)$ want $\underline{r}(s^{-1}(t))$

(why: ~~start~~ arc length of $\underline{r}(s^{-1}(t))$ is $s(s^{-1}(t)) = t$)

<u>Example</u> $s^{-1}(t) = \frac{t}{3}$.

③ write down reparameterized curve.

$$\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t)) = \underline{r}(t/3) = \left\langle \tfrac{2}{3}t, 1 - \tfrac{2}{3}t, \tfrac{1}{3}t \right\rangle.$$

check : $\|\underline{r}(t)\| = \| \langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \rangle \| = \sqrt{\frac{4}{9}+\frac{4}{9}+\frac{1}{9}} = 1.$

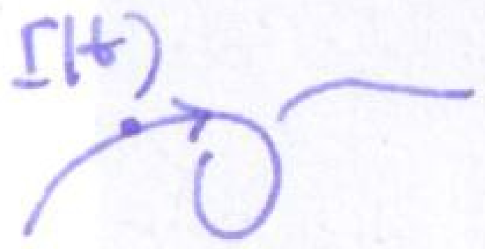
Summary $\underline{r}(t)$ arbitrary parameterization

$s(t)$ arc length $= \int_0^t \|\underline{r}'(u)\|\, du$

$s^{-1}(t)$ inverse function of s.

then the arc length parameterization is $\hat{\underline{r}}(t) = \underline{r}(s^{-1}(t))$.

§13.5 Motion in $\mathbb{R}^3$



location / position $\underline{r}(t)$

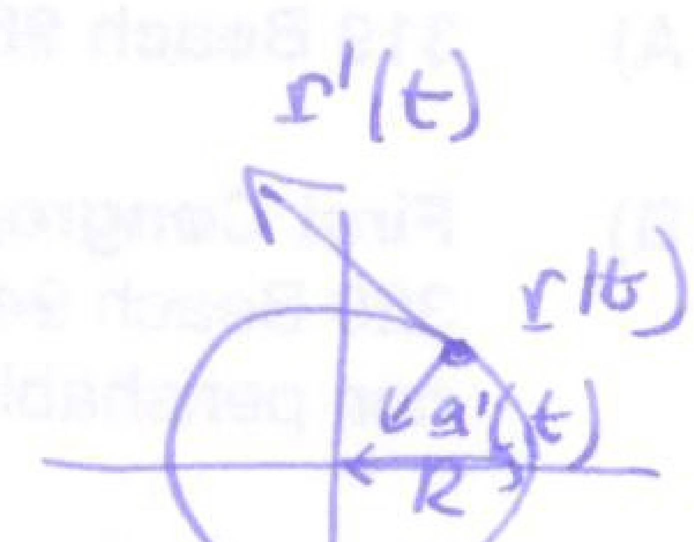
velocity $\underline{v}(t) = \underline{r}'(t)$ speed $\|\underline{r}'(t)\| = \|\underline{v}(t)\|$

acceleration $\underline{a}(t) = \underline{r}''(t)$.

Example suppose position given by $\underline{r}(t) = \langle e^{-t}, \ln(t), \sqrt{t} \rangle$

then velocity $\underline{v}(t) = \underline{r}'(t) = \langle -e^{-t}, \frac{1}{t}, \frac{1}{2}t^{-1/2} \rangle$

acceleration $\underline{a}(t) = \underline{r}''(t) = \langle e^{-t}, -\frac{1}{t^2}, -\frac{1}{4}t^{-3/2} \rangle$



Example Acceleration for uniform circular motion

$\underline{r}(t) = \langle R\cos(\omega t), R\sin(\omega t) \rangle$ ω constant called angular velocity

If speed is constant v, then $\|\underline{r}'(t)\| = \| \langle -R\omega \sin(\omega t), -R\omega\cos(\omega t) \rangle \|$

$= \sqrt{R^2\omega^2(\sin^2(\omega t) + \cos^2(\omega t))} = R\omega$ so $v = R\omega \Leftrightarrow \omega = \frac{v}{R}$

acceleration $\underline{a}(t) = \underline{r}''(t) = \langle -R\omega^2\cos(\omega t), -R\omega^2\sin(\omega t) \rangle$

so $\|\underline{a}''(t)\| = R\omega^2 = \frac{Rv^2}{R^2} = \frac{v^2}{R}$

so the acceleration vector $\underline{a}(t)$ has constant length $\frac{v^2}{R}$ and always points towards the origin.