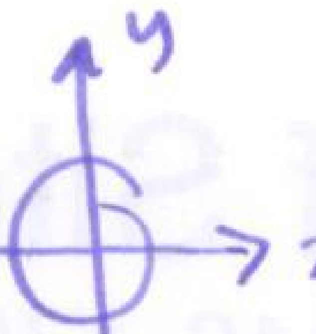
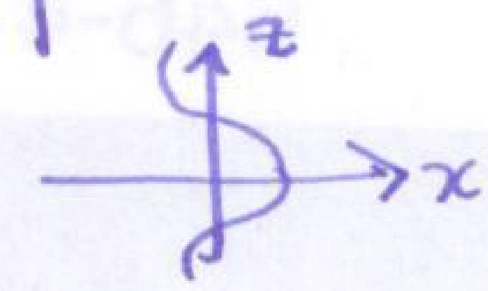
projections : set one cordinate zero

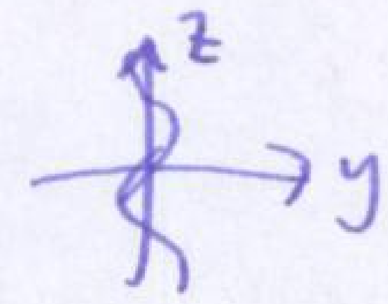
$\langle \cos(t), \sin(t), 0 \rangle$ projection onto xy-plane.



$\langle \cos(t), 0, t \rangle$ xz-plane

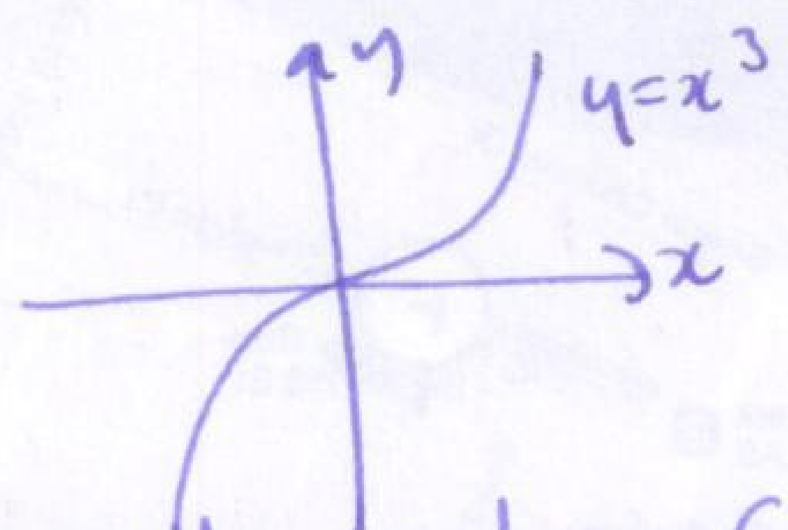


$\langle 0, \sin(t), t \rangle$ yz-plane



Finding parameterizations

Example ① $y = x^3$



$\underline{r}(t) = \langle t, t^3 \rangle$

② circle of radius 4 in xz-plane with center $(1,2,3)$.

unit circle in xy-plane : $\underline{r}(t) = \langle \cos(t), \sin(t), 0 \rangle$

xz-plane : $\underline{r}(t)$ $\langle \cos(t), 0, \sin(t) \rangle$

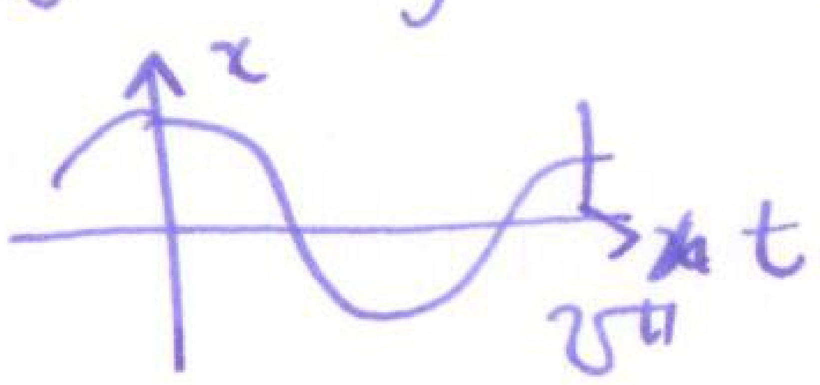
radius 4 : $\underline{r}(t) = \langle 4\cos(t), 0, 4\sin(t) \rangle$

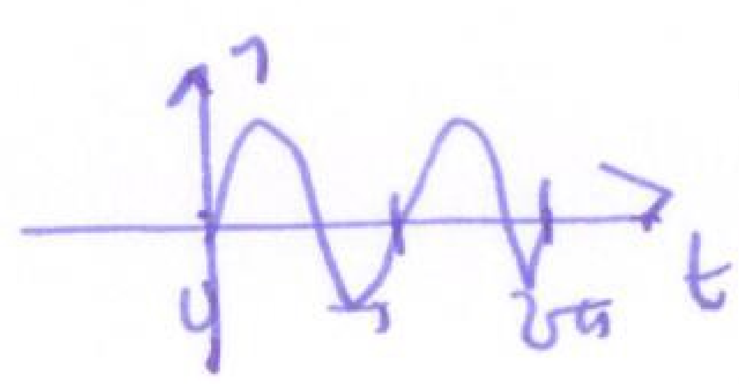
translate : $\underline{r}(t) = \langle 4\cos(t)+1, 2, 4\sin(t)+3 \rangle$

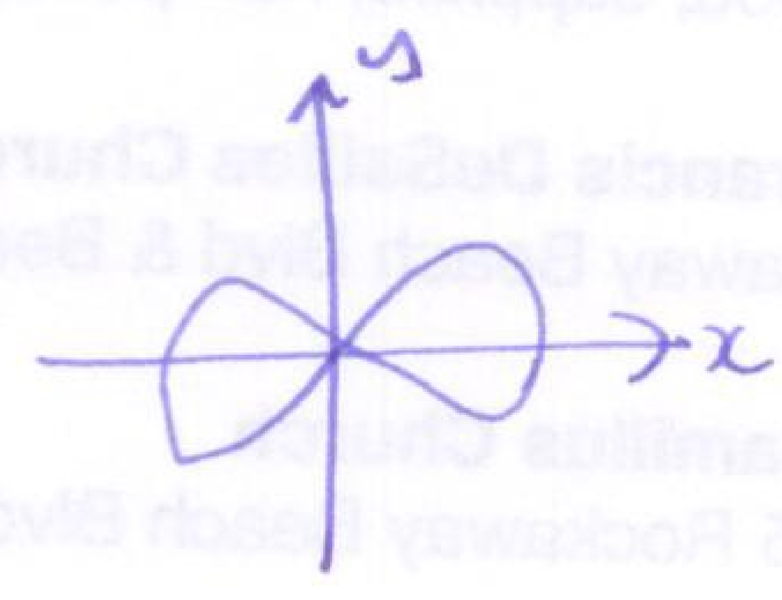
Drawing parametrized curves

Example $\underline{r}(t) = \langle \cos(t), \sin(2t) \rangle$

try drawing (x,t), (y,t) pictures.







§13.2 Calculus of vector valued functions

Limits Defⁿ A vector valued function $\underline{r}(t)$ has a limit $\underline{v}$ at t_0 if

$\lim_{t \to t_0} \| \underline{r}(t) - \underline{v} \| = 0$. If this happens we ~~say~~ write $\lim_{t \to t_0} \underline{r}(t) = \underline{v}$

Thm 1 Limits are computed componentwise.

$\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ has a limit as $t \to t_0$

$\Leftrightarrow$ each component function has a limit, and

$$\lim_{t \to t_0} \underline{r}(t) = \left\langle \lim_{t \to t_0} x(t), \lim_{t \to t_0} y(t), \lim_{t \to t_0} z(t) \right\rangle$$

Proof (sketch) $\|\underline{r}(t) - \underline{v}\|^2 = (x(t) - v_1)^2 + (y(t) - v_2)^2 + (z(t) - v_3)^2$

LHS $\to 0 \Rightarrow$ each term on RHS $\to 0$. $\square$

Example find $\lim_{t \to 0} \underline{r}(t)$ where $\underline{r}(t) = \langle t^2, t, \frac{\sin t}{t} \rangle$

$$\lim_{t \to 0} \underline{r}(t) = \left\langle \lim_{t \to 0} t^2, \lim_{t \to 0} t, \lim_{t \to 0} \frac{\sin(t)}{t} \right\rangle = \langle 0, 0, 1 \rangle.$$

Continuity

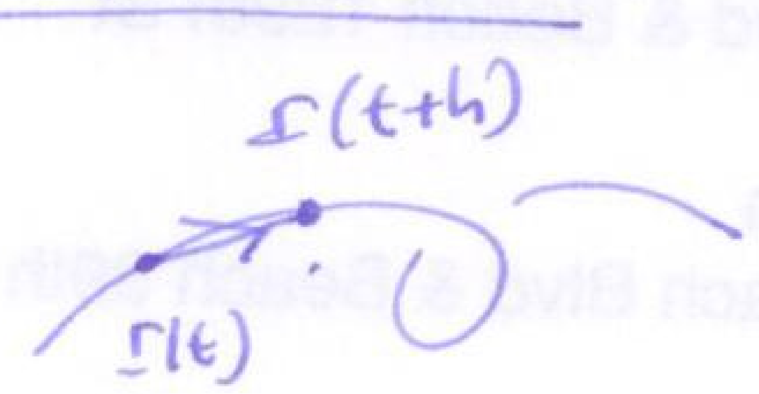
Def^{n} $\underline{r}(t)$ is continuous $\Leftrightarrow$ all of its components are continuous.

Differentiation

Def^{n} The derivative of $\underline{r}(t)$ is $\underline{r}'(t) = \frac{d\underline{r}}{dt} = \lim_{h \to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$

if this limit exists we say $\underline{r}(t)$ is differentiable

Remark $\frac{\underline{r}(t+h) - \underline{r}(t)}{h}$ makes sense!



Thm 2 Derivatives are computed componentwise.

if $\underline{r}(t) = \langle x(t), y(t), z(t) \rangle$ then $\underline{r}'(t) = \langle x'(t), y'(t), z'(t) \rangle$

$$\underline{r}'(t) = \frac{d\underline{r}}{dt}$$

Proof (sketch) derivative defined as a limit, limits computed componentwise $\square$.

Example $\underline{r}(t) = \langle \cos t, \sin t, t \rangle$

$\underline{r}'(t) = \langle -\sin t, \cos t, 1 \rangle$

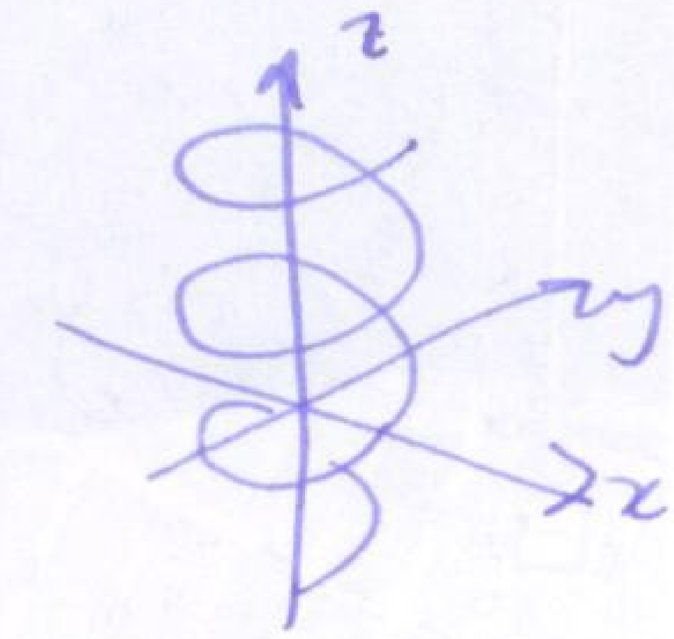
$\underline{r}(t) = \langle e^{2t}, t\sin t, \frac{\cos t}{t} \rangle$.

$\underline{r}'(t) = \langle 2e^{2t}, \sin t + t\cos t, \frac{\quad}{t^2} \rangle$

Higher derivatives

$$\underline{r}''(t) = \frac{d^2 \underline{r}}{dt^2} = \lim_{h \to 0} \frac{\underline{r}'(t+h) - \underline{r}'(t)}{h}.$$

Example $\underline{r}''(t) = \langle -\cos t, -\sin t, 0 \rangle$



if $\underline{r}(t)$ is position

then $\underline{r}'(t)$ is velocity

$\underline{r}''(t)$ is acceleration.

Differentiation rules (assume all functions differentiable)

- sums: $(\underline{r}_1(t) + \underline{r}_2(t))' = \underline{r}_1'(t) + \underline{r}_2'(t)$
- scalar mult: $(c\,\underline{r}(t))' = c\,\underline{r}'(t)$ c constant (does not depend on t)
- product rule: if $f(t)$, differentiable scalar function then

$$(f(t)\underline{r}(t))' = f(t)\underline{r}'(t) + f'(t)\underline{r}(t)$$

- chain rule: $\frac{d}{dt}\left(\underline{r}(g(t))\right) = g'(t)\,\underline{r}'(g(t))$

Warning: $g(\underline{r}(t))$ doesn't make sense!

$\underline{r}_1(\underline{r}_2(t))$ "   " .

Proof (sketch) everything is defined componentwise.

product rule: $f(t)\underline{r}(t) = f(t)\langle x(t), y(t), z(t) \rangle = \langle f(t)x(t), f(t)y(t), f(t)z(t) \rangle$

$(f(t)\underline{r}(t))' = \langle f'x + fx', f'y + fy', f'z + fz' \rangle = f'\langle x, y, z \rangle + f\langle x', y', z' \rangle$

$= f'(t)\underline{r}(t) + f(t)\underline{r}'(t)$ $\square$.

Thm 3 Product rule for dot and cross products.

dot product: $(\underline{r}_1(t)\cdot \underline{r}_2(t))' = \underline{r}_1(t)\cdot \underline{r}_2'(t) + \underline{r}_1'(t)\,\underline{r}_2(t)$

cross product: $(\underline{r}_1(t)\times \underline{r}_2(t))' = \underline{r}_1(t)\times \underline{r}_2'(t) + \underline{r}_1'(t)\times \underline{r}_2(t)$

warning: order matters in the cross product!

Proof (sketch) dot, cross products defined componentwise.

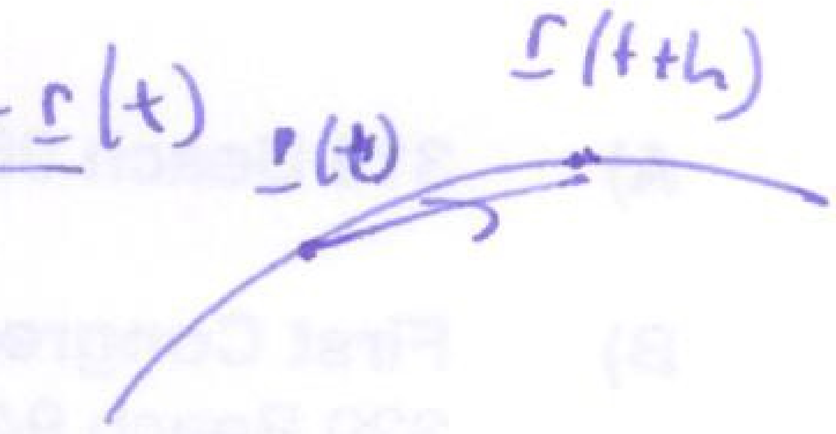
simple example in $\mathbb{R}^2$. $\underline{r}_1(t)\cdot\underline{r}_2(t) = \langle x_1(t), y_1(t)\rangle \cdot \langle x_2(t), y_2(t)\rangle$

$$(\underline{r}_1(t)\cdot \underline{r}_2(t))' = (x_1x_2 + y_1y_2)' = x_1'x_2 + x_1x_2' + y_1'y_2 + y_1y_2'$$

$$= x_1'x_2 + y_1'y_2 + x_1x_2' + y_1y_2' = \underline{r}_1'(t)\cdot\underline{r}_2(t) + \underline{r}_1(t)\cdot\underline{r}_2'(t). \square$$

The derivative is the tangent vector (or velocity vector)

$$\underline{r}'(t) = \lim_{h\to 0} \frac{\underline{r}(t+h) - \underline{r}(t)}{h}$$



tangent line: is given by $\underline{L}(t) = \underline{r}(t_0) + t\,\underline{r}'(t_0)$

Example find tangent line to helix $\underline{r}(t) = \langle \cos t, \sin t, t\rangle$ at $t=1$

Example show $\underline{r}(t)$ and $\underline{r}'(t)$ are orthogonal if $\underline{r}(t)$ has unit length

unit length: $\|\underline{r}(t)\|^2 = 1$

$$\| \quad \underline{r}(t)\cdot\underline{r}(t)$$

so $(\underline{r}(t)\cdot\underline{r}(t))' = \underline{r}'(t)\cdot\underline{r}(t) + \underline{r}(t)\cdot\underline{r}'(t) = 2\underline{r}'(t)\cdot\underline{r}(t) = 0.$

$\Rightarrow \underline{r}(t) \perp \underline{r}'(t)$

Geometric explanation: