§12.3 Dot product and angle

$\underline{v}, \underline{w}$ two vectors.

Dot product

$\underline{v}\cdot\underline{w}$ geometric def$^{\underline{n}}$: $\|\underline{v}\|\|\underline{w}\|\cos\theta$ $\quad\theta$ angle between vectors

coordinate def$^{\underline{n}}$: $\underline{v} = \langle v_1, v_2, v_3\rangle \quad \underline{w} = \langle w_1, w_2, w_3\rangle$

$$\underline{v}\cdot\underline{w} = v_1w_1 + v_2w_2 + v_3w_3$$

useful properties

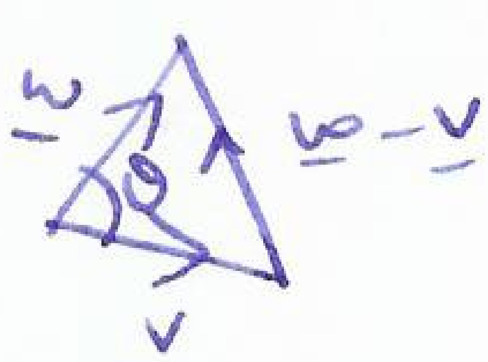
- $\underline{0}\cdot\underline{v} = \underline{v}\cdot\underline{0} = 0$
- $\underline{v}\cdot\underline{w} = \underline{w}\cdot\underline{v}$ (commutativity)
- $(\lambda\underline{v})\cdot\underline{w} = \underline{v}\cdot(\lambda\underline{w}) = \lambda(\underline{v}\cdot\underline{w})$ (scalar dist.)
- $\underline{u}\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v} + \underline{u}\cdot\underline{w}$ (dist.)
- $\underline{v}\cdot\underline{v} = \|\underline{v}\|^2$ (length)

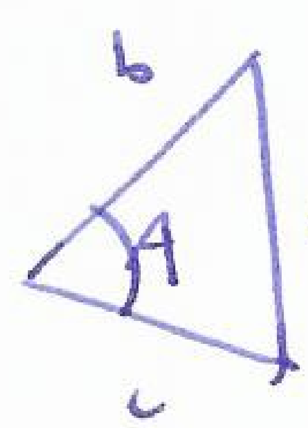
Angle between two vectors

$$\underline{v}\cdot\underline{w} = \|\underline{v}\|\,\|\underline{w}\|\cos\theta$$

$$\cos\theta = \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\,\|\underline{w}\|}$$

Proof (law of cosines)





$$a^2 = b^2 + c^2 - 2ab\cos A$$

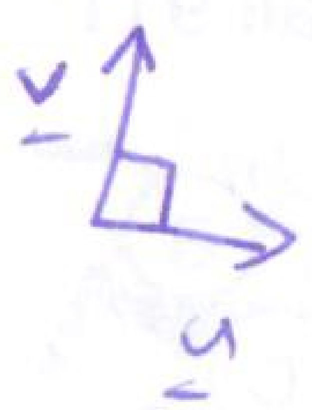
$$\|\underline{w}-\underline{v}\|^2 = \|\underline{w}\|^2 + \|\underline{v}\|^2 - 2\|\underline{v}\|\|\underline{w}\|\cos\theta$$

$$\|\underline{w}-\underline{v}\|^2 = (\underline{w}-\underline{v})\cdot(\underline{w}-\underline{v}) = \underline{w}\cdot\underline{w} - \underline{w}\cdot\underline{v} - \underline{v}\cdot\underline{w} + \underline{v}\cdot\underline{v}$$
$$= \|\underline{w}\|^2 - 2\underline{w}\cdot\underline{v} + \|\underline{v}\|^2.$$

$$\cancel{\|\underline{w}\|^2} - 2\underline{w}\cdot\underline{v} + \cancel{\|\underline{v}\|^2} = \cancel{\|\underline{w}\|^2} + \cancel{\|\underline{v}\|^2} - 2\|\underline{w}\|\|\underline{v}\|\cos\theta. \Rightarrow \cos\theta = \frac{\underline{v}\cdot\underline{w}}{\|\underline{v}\|\|\underline{w}\|} \quad \square.$$

Def^{n} $\underline{u}, \underline{v}$ are perpendicular (or orthogonal) iff $\underline{u} \cdot \underline{v} = 0$

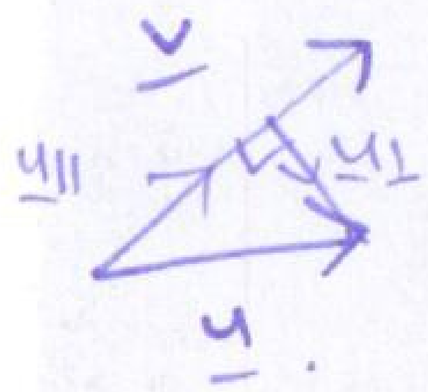
($\underline{u} \perp \underline{v}$)



Projections given vectors $\underline{u}, \underline{v}$ we can write $\underline{u}$ as

$\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$ where $\underline{u}_{\parallel}$ is parallel to $\underline{v}$

$\underline{u}_{\perp}$ is perpendicular to $\underline{v}$



$\underline{u}_{\parallel}$ is called the projection of $\underline{u}$ to $\underline{v}$, also written $\text{proj}_{\underline{v}}(\underline{u})$

Q: what is $\underline{u}_{\parallel}$?

A: $\underline{u}_{\parallel} = c\underline{v}$ for some number c

$$\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$$

$$\underline{u} = c\underline{v} + \underline{u}_{\perp}$$

take dot product with $\underline{v}$.

$$\underline{u} \cdot \underline{v} = c\underline{v} \cdot \underline{v} + \underbrace{\underline{u}_{\perp} \cdot \underline{v}}_{=0}$$

$= 0$ as $\underline{v}, \underline{u}_{\perp}$ are perpendicular

$\Rightarrow$ $c = \dfrac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}}$ so $\underline{u}_{\parallel} = \dfrac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v}$.

Summary given $\underline{u}, \underline{v}$ $\underline{u} = \underline{u}_{\parallel} + \underline{u}_{\perp}$

where $\underline{u}_{\parallel} = \text{proj}_{\underline{v}}(\underline{u}) = \dfrac{\underline{u} \cdot \underline{v}}{\underline{v} \cdot \underline{v}} \underline{v}$ and $\underline{u}_{\perp} = \underline{u} - \text{proj}_{\underline{v}}(\underline{u})$.

Example $\underline{u} = \langle 1,2,3 \rangle$ $\quad$ $\underline{u}_{\parallel} = \text{proj}_{\underline{v}}(\underline{u}) = \dfrac{\underline{u}\cdot\underline{v}}{\underline{v}\cdot\underline{v}}\,\underline{v} = \dfrac{6}{3}\langle 1,1,1 \rangle$

$\underline{v} = \langle 1,1,1 \rangle$ $\quad$ $= \langle 2,2,2 \rangle$

$$\underline{u}_{\perp} = \underline{u} - \text{proj}_{\underline{v}}(\underline{u})$$
$$= \langle 1,2,3 \rangle - \langle 2,2,2 \rangle = \langle -1,0,1 \rangle.$$

check: $\underline{u}_{\perp}\cdot\underline{v} = 0$ $\quad$ $\langle -1,0,1 \rangle \cdot \langle 1,1,1 \rangle = 0.$

§12.4 Cross product

dot product $\underline{v}\cdot\underline{w}$ ← scalar!

cross product $\underline{v}\times\underline{w}$ ← vector!

Recall: A 2x2 matrix $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ then $\det(A) = a_{11}a_{22} - a_{12}a_{21}$

Example $\det\left(\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\right) = 2-1 = 1$

A 3x3 matrix: $\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

Def^n if $\underline{v} = \langle v_1, v_2, v_3 \rangle$ $\quad$ $\underline{w} = \langle w_1, w_2, w_3 \rangle$

then $\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \underline{i}\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \underline{j}\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \underline{k}\begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$

$$= \langle v_2w_3 - v_3w_2,\ -v_1w_3 + w_1v_3,\ v_1w_2 - v_2w_1 \rangle.$$

Example $\underline{v} = \langle 1,2,3 \rangle$ $\quad$ $\underline{w} = \langle 1,-1,-1 \rangle$

$\underline{v}\times\underline{w} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 1 & 2 & 3 \\ 1 & -1 & -1 \end{vmatrix} = \underline{i}\begin{vmatrix} 2 & 3 \\ -1 & -1 \end{vmatrix} - \underline{j}\begin{vmatrix} 1 & 3 \\ 1 & -1 \end{vmatrix} + \underline{k}\begin{vmatrix} 1 & 2 \\ 1 & -1 \end{vmatrix} = \langle 1, 5, 1 \rangle$

Thm (Geometric def^n of cross product)

$\underline{v} \times \underline{w}$ is the unique vector which is

1) perpendicular to both $\underline{v}$ and $\underline{w}$
2) $\|\underline{v} \times \underline{w}\| = \|\underline{v}\|\|\underline{w}\||\sin\theta|$ (θ angle between $\underline{v}, \underline{w}$)
3) $\{\underline{v}, \underline{w}, \underline{v} \times \underline{w}\}$ is <u>right handed</u>

<u>Warning</u> $\underline{v} \times \underline{w} = -\underline{w} \times \underline{v}$ (anti-commutative!)

<u>Note</u>: $\underline{v} \times \underline{v} = \underline{0} = -\underline{v} \times \underline{v}$.

Thm Useful properties:

- $\underline{w} \times \underline{v} = -\underline{v} \times \underline{w}$
- $\underline{v} \times \underline{v} = \underline{0}$
- $\underline{v} \times \underline{w} = \underline{0} \Leftrightarrow \underline{w} = \lambda \underline{v}$ for some $\lambda \in \mathbb{R}$
- $(\lambda \underline{v}) \times \underline{w} = \underline{v} \times (\lambda \underline{w}) = \lambda(\underline{v} \times \underline{w})$
- $(\underline{u} + \underline{v}) \times \underline{w} = \underline{u} \times \underline{w} + \underline{v} \times \underline{w}$
- $\underline{u} \times (\underline{v} + \underline{w}) = \underline{u} \times \underline{v} + \underline{u} \times \underline{w}$

<u>special case</u>

$\underline{i} \times \underline{j} = \underline{k}$ $\quad \underline{j} \times \underline{i} = -\underline{k}$

$\underline{j} \times \underline{k} = \underline{i}$ $\quad \underline{k} \times \underline{j} = -\underline{i}$

$\underline{k} \times \underline{i} = \underline{j}$ $\quad \underline{i} \times \underline{k} = -\underline{j}$

Alternate way of computing $\underline{v} \times \underline{w}$:

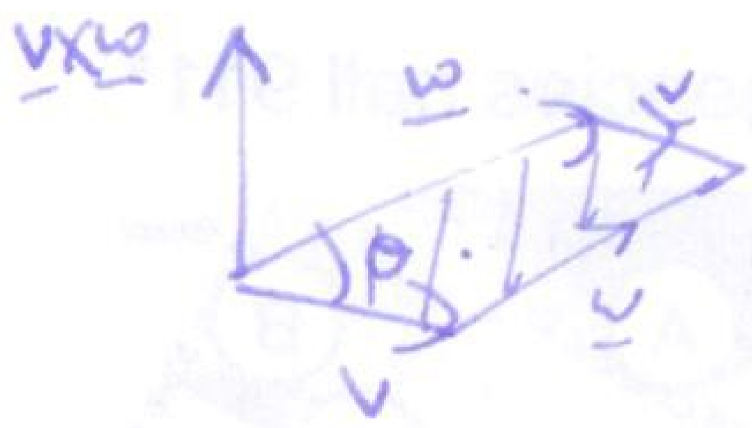
$\underline{v} = \langle 1, 0, 1 \rangle = \underline{i} + \underline{k}$

$\underline{w} = \langle 0, -1, 0 \rangle = -\underline{j}$

then $\underline{v} \times \underline{w} = (\underline{i} + \underline{k}) \times (-\underline{j})$

$= -\underline{i} \times \underline{j} - \underline{k} \times \underline{j} = -\underline{k} + \underline{i}$.

useful facts



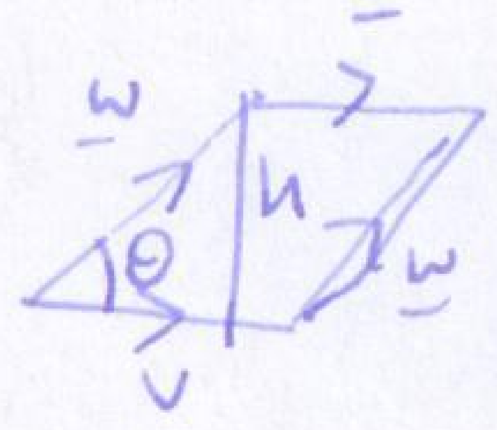
area of parallelogram determined by $\underline{v}, \underline{w}$ is

$$\text{Area} \quad A = \| \underline{v} \times \underline{w} \|.$$

$\underline{u}, \underline{v}, \underline{w}$ determine a parallelopiped

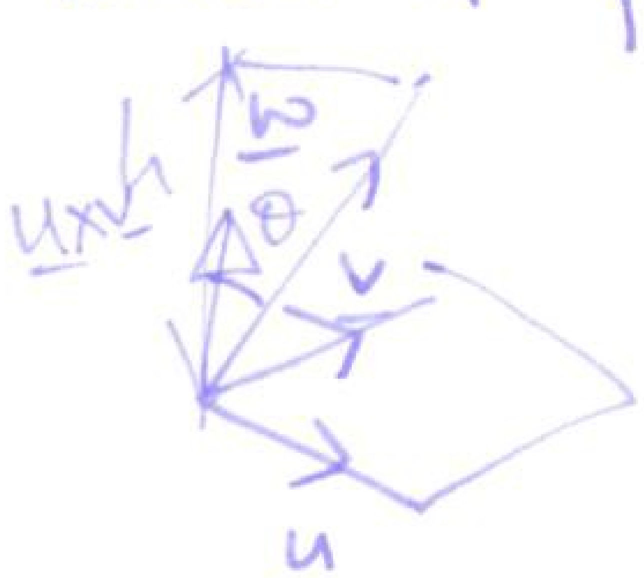
$$\text{volume} \quad V = | \underline{u} \cdot (\underline{v} \times \underline{w}) | \quad \text{(order doesn't matter).}$$

check



$$\begin{aligned} \text{area} &= \text{base} \times \text{height} \\ &= \|\underline{v}\| . h \\ &= \|\underline{v}\| \, \|\underline{w}\| \sin\theta . \end{aligned}$$

volume of parallelopiped is = area of base × height



$$\begin{aligned} & \left| \, \| \underline{u} \times \underline{v} \| \, \|\underline{w}\| \cos\theta \, \right| \\ &= | \, \underline{w} \cdot (\underline{u} \times \underline{v}) \, | \end{aligned}$$

Notation $\underline{u} . (\underline{v} \times \underline{w})$ is called the triple vector product

Warning $\underline{u} . (\underline{v} \times \underline{w})$ makes sense

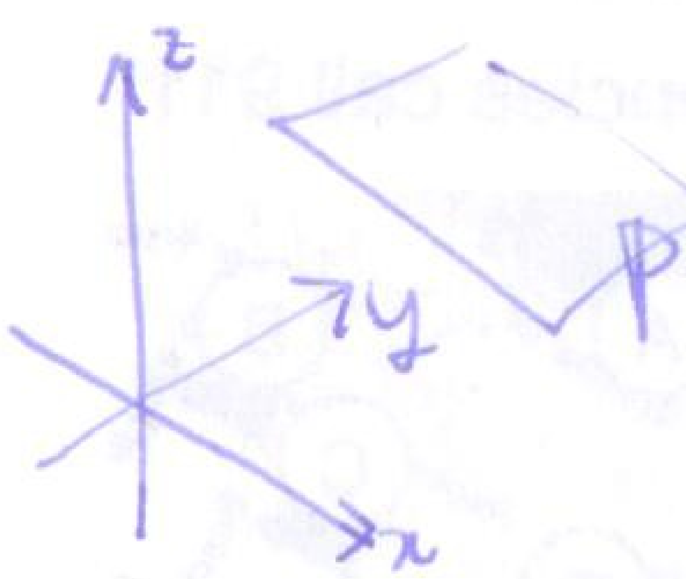
$(\underline{u} . \underline{v}) \times \underline{w}$ does not!

Note if $\underline{u} = \langle u_1, u_2, u_3 \rangle$ then $\underline{u} . (\underline{v} \times \underline{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

$\underline{v} = \langle v_1, v_2, v_3 \rangle$

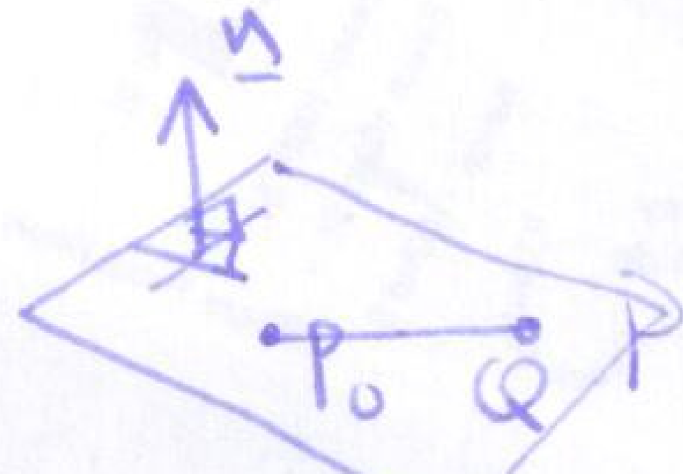
$\underline{w} = \langle w_1, w_2, w_3 \rangle$

§12.5 Planes in $\mathbb{R}^3$



Claim: a plane P in $\mathbb{R}^3$ is defined by a single linear equation $ax+by+cz=d$

Proof:



let P_0 be a point on P $\quad P_0 = (x_0, y_0, z_0)$

let $\underline{n}$ be the normal vector to P $\quad \underline{n} = \langle a, b, c \rangle$

let Q be some other point on P: then $\quad \overrightarrow{P_0Q} \cdot \underline{n} = 0.$

$Q = (x, y, z)$ so $\quad \overrightarrow{P_0Q} = \langle x - x_0, y - y_0, z - z_0 \rangle$

$\overrightarrow{P_0Q} \cdot \underline{n} = 0 \qquad \langle x - x_0, y - y_0, z - z_0 \rangle \cdot \langle a, b, c \rangle = 0$

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

$$ax + by + cz = \underbrace{ax_0 + by_0 + cz_0}_{d}$$

Summary

$ax+by+cz = d$ is equation of a plane in scalar form

$$\left.\begin{array}{l} \overrightarrow{P_0Q} \cdot \underline{n} = 0 \\ \text{or } (\underline{x} - \underline{p}) \cdot \underline{n} = 0 \end{array}\right\} \text{equation of plane in vector form}$$

Example find equation of plane through $P_0 = (1, 2, 3)$ with normal vector $\underline{n} = \langle 2, 1, -1 \rangle$

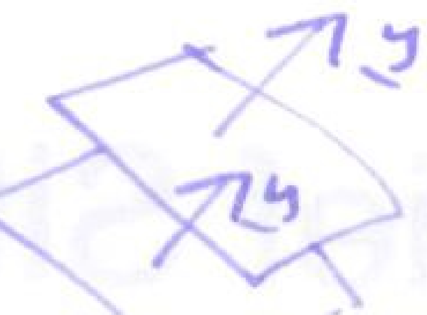
$\underline{n} \cdot \overrightarrow{P_0Q} = 0 \qquad \langle 2, 1, -1 \rangle \cdot \langle x-1, y-2, z-3 \rangle = 0$

$$2x - 2 + y - 2 - z + 3 = 0$$

$$2x + y - z = 1$$

Observations • $ax+by+cz=d$ has normal vector $\underline{n} = \langle a, b, c \rangle$.

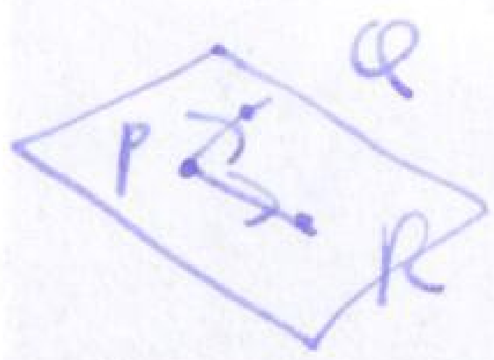
- parallel planes have the same normal vector



Example: find the plane parallel to $x-3y+2z=4$ through the point $P=(6,4,2)$ $n=\langle 1,-3,2\rangle$

$(\underline{x}-\underline{P})\cdot\underline{n}=0$ $(x-6)\cancel{+}-3(y-4)+2(z-2)=0.$

- three points determine a plane



find normal vector $\underline{n}=\overrightarrow{PQ}\times\overrightarrow{PR}$.

Example $P=(1,0,1)$ $Q=(2,3,1)$ $R=(-1,-1,3)$

$\overrightarrow{PQ}=\langle 1,3,0\rangle$ $\overrightarrow{PR}=\langle -2,-1,2\rangle$

$$\underline{n}=\overrightarrow{PQ}\times\overrightarrow{PR}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ 1 & 3 & 0\\ -2 & -1 & 2\end{vmatrix}=\underline{i}\begin{vmatrix}3 & 0\\ -1 & 2\end{vmatrix}-\underline{j}\begin{vmatrix}1 & 0\\ -2 & 2\end{vmatrix}+\underline{k}\begin{vmatrix}1 & 3\\ -2 & -1\end{vmatrix}$$

$$=\langle 6,-2,-7\rangle$$

equation: $6(x-1)-2y-7(z-1)=0$

- find intersection of a plane and a line

plane: $2x-3y+4z=2$

line: $\langle 1,2,1\rangle+t\langle -2,1,1\rangle$ $x=1-2t$
$y=2+t$
$z=1+t$

substitute in to equation of plane:

solve for t: $2(1-2t)\cancel{+}-3(2+t)+4(1+t)=2$

$2-4t-6-3t+4+4t=2$

$-3t=2$ $t=-2/3$.

so point is $\langle 1+\frac{4}{3}, 2\cancel{4}-\frac{2}{3}, 1-\frac{2}{3}\rangle=\langle\frac{7}{3},\frac{4}{3},\frac{1}{3}\rangle$.

check!