Math 233 Calculus 3

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- Math tutoring 1S-214
- students with disabilities.

Text: Calculus (Early transcendentals) Rogawski.

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§12.1 2d vectors

scalar/number

size/magnitude only

examples 7
-4.3 etc.

examples: length
temperature
pressure
time
speed = length of velocity vector

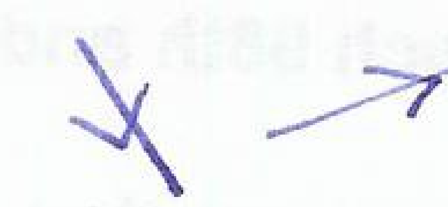
notation: $7, \pi \in \mathbb{R}$

$s, t \in \mathbb{R}$

vector

size and direction

examples

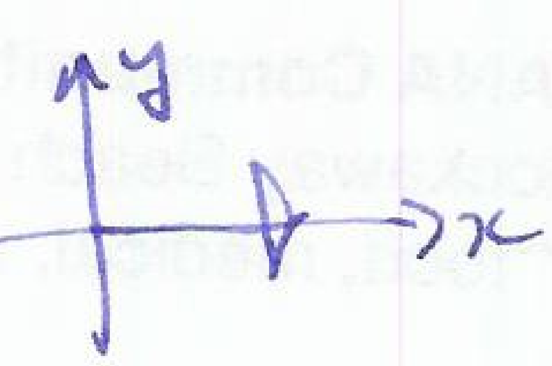


length and direction

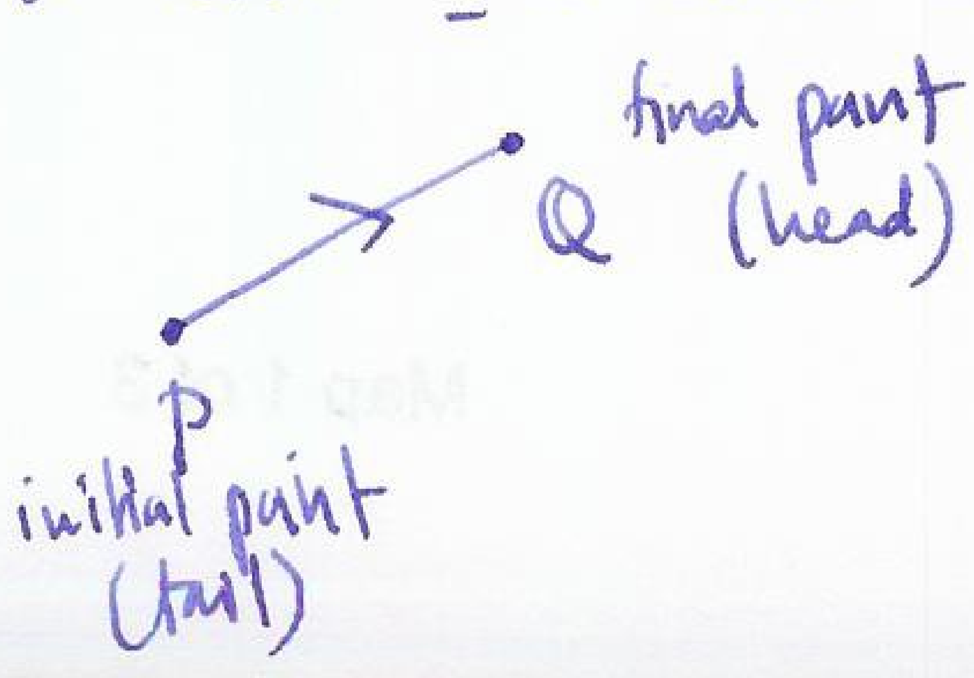
examples: force
velocity

notation: $\underline{v}$ $\vec{v}$

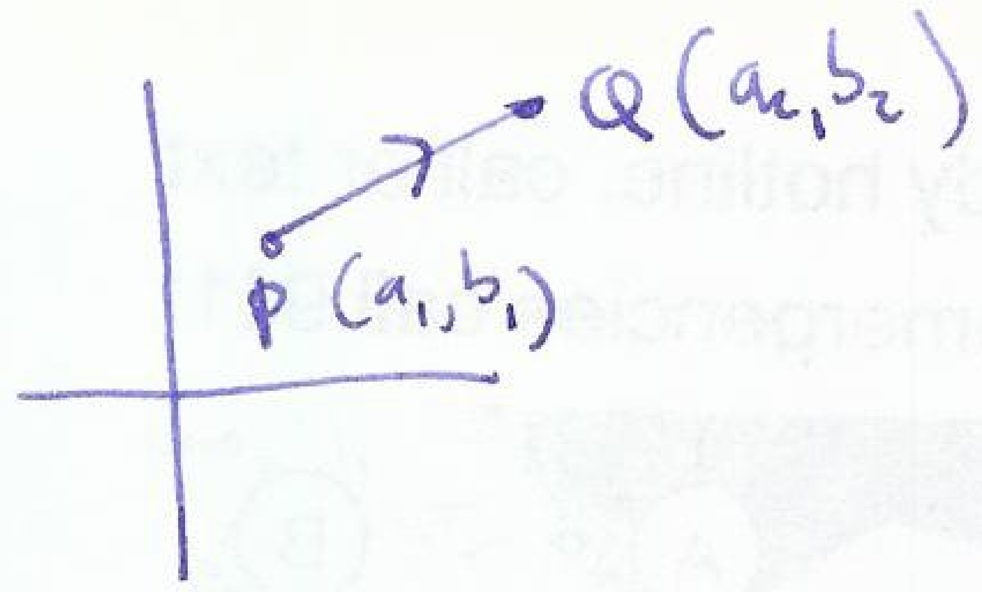
"length 4 in direction of x-axis"



a vector $\underline{v}$ is determined by its initial and final points



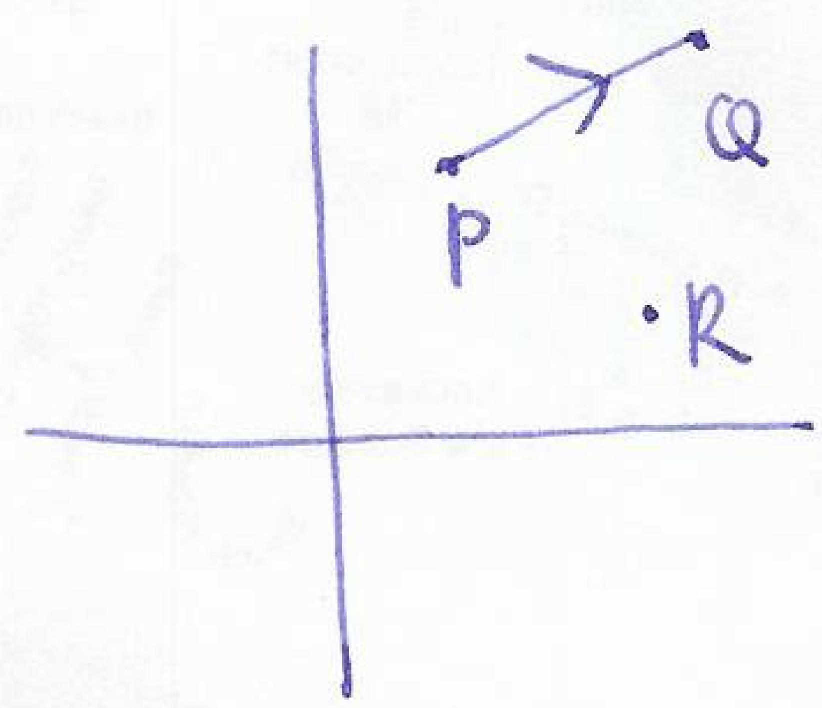
notation $\underline{v} = \vec{v} = \overrightarrow{PQ}$



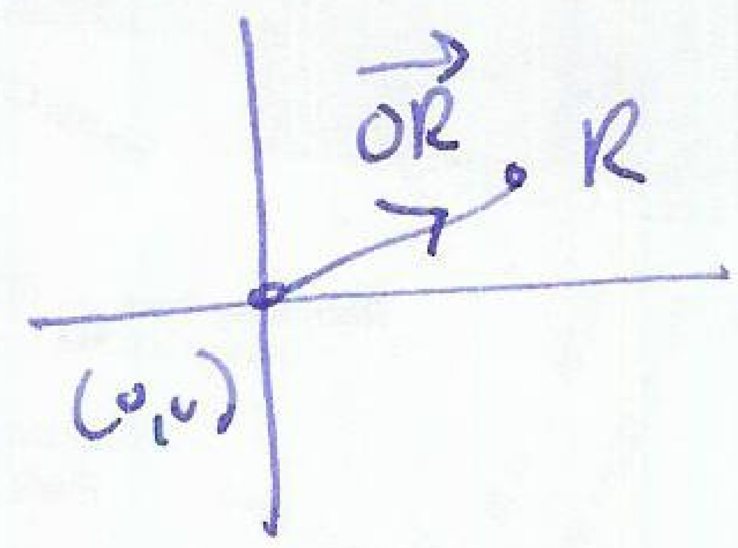
$\underline{Q}$: how long is the vector?

$\underline{A}$: $\|\underline{v}\| = \|\overrightarrow{PQ}\| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2}$

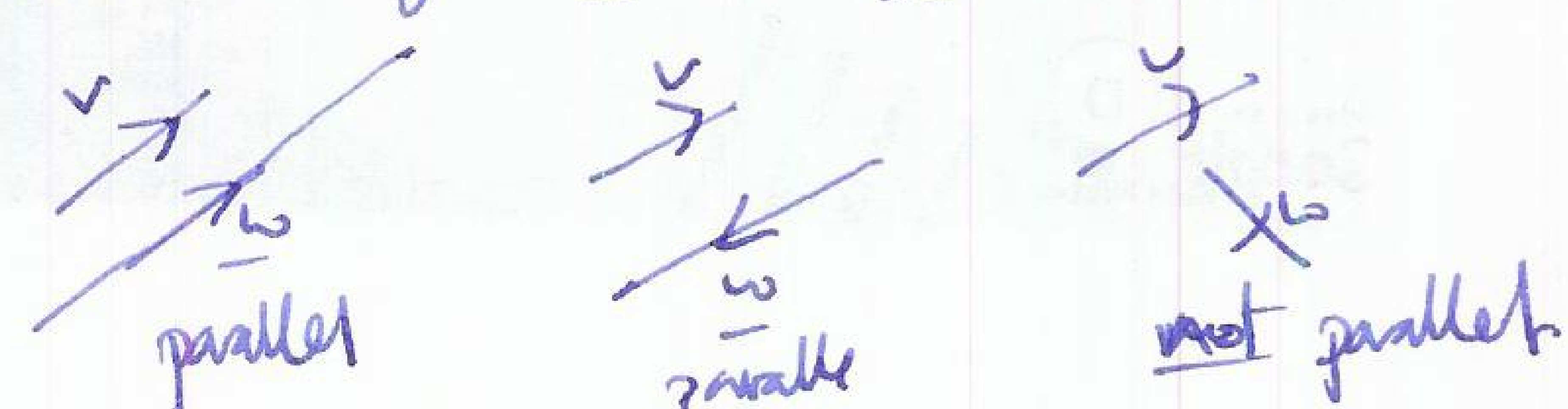
vectors vs points



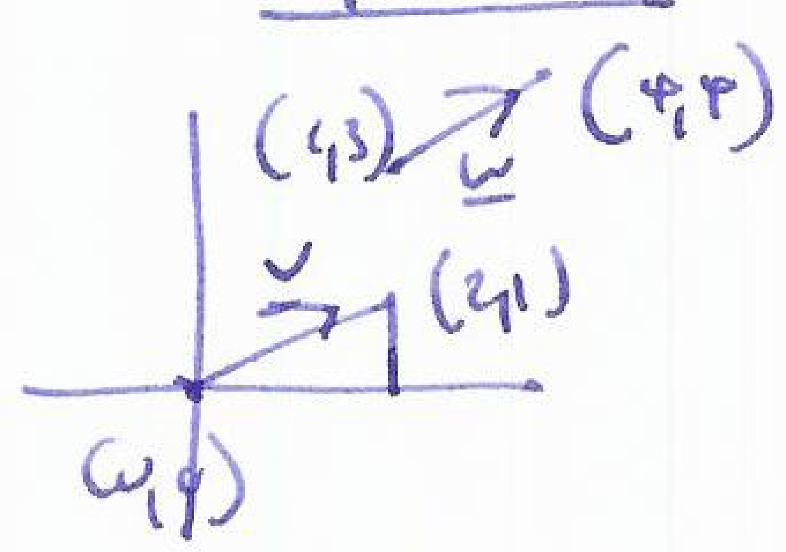
every point corresponds to a special <u>position vector</u> from $O = (0,0)$ to R.



- two vectors $\underline{v}, \underline{w}$ are <u>parallel</u> if the lines through $\underline{v}$ and $\underline{w}$ have the same direction (or opposite direction)



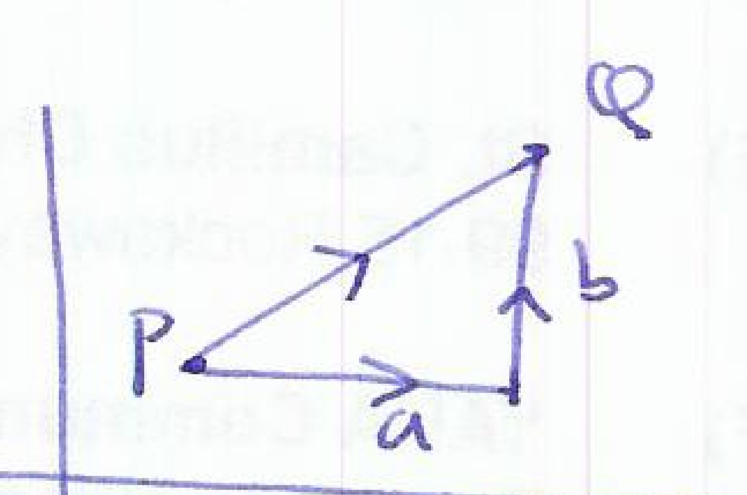
- two vectors $\underline{v}, \underline{w}$ are <u>translates</u> or <u>equivalent</u> or <u>equal</u> if they have the same length and direction



$\underline{v} = \underline{w}$.

<u>observation</u>: every vector $\underline{v}$ is equal to a unique position vector $\underline{v}_0$ based at the origin $O := (0,0)$.

<u>Def^n</u> <u>components</u> of a vector $\underline{v} = \overrightarrow{PQ}$



$P = (a_1, b_1)$ $\quad Q = (a_2, b_2)$

are $\underline{v} = \langle a, b \rangle$ $\quad a = a_2 - a_1$ $\quad b = b_2 - b_1$

x-component $\quad$ y-component

<u>observation</u> $\|\underline{v}\| = \sqrt{a^2+b^2}$!

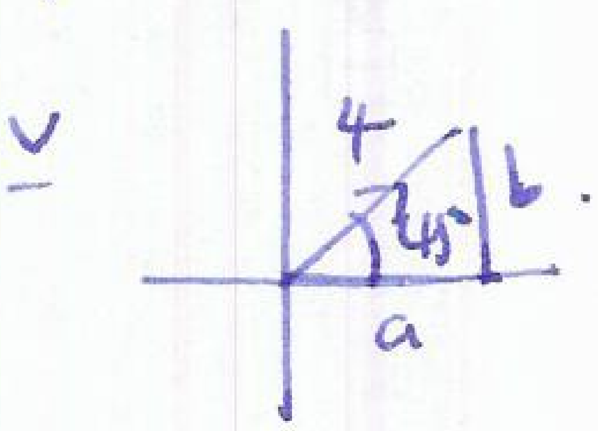
- the components $\langle a,b\rangle$ determine length and directions so two vectors are equal $\Leftrightarrow$ they have the same components.

- $\underline{v} = \langle a,b\rangle$ does not determine basepoint.

<u>convention</u> all vectors based at origin O unless otherwise stated

<u>special vector</u> $\underline{0} = \langle 0,0\rangle$ <u>zero vector</u>

<u>Example</u> A vector $\underline{v}$ has length 4, lies in the 1st quadrant, and makes an angle of $\pi/4$ with the x-axis. Find the components of $\underline{v}$

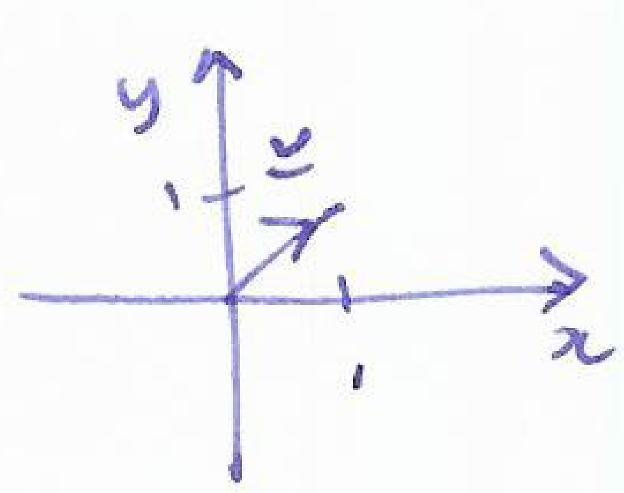


<u>Vector addition and scalar multiplication</u>

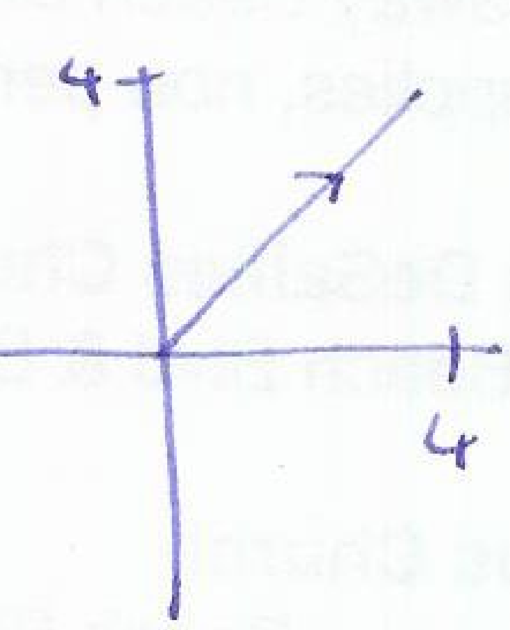
<u>scalar multiplication</u> λ scalar number
$\underline{v}$ vector

$\lambda\underline{v}$ is the vector with same direction as $\underline{v}$ but length $\lambda\|\underline{v}\|$ if $\lambda>0$
" opposite " " $|\lambda|\|\underline{v}\|$ if $\lambda<0$

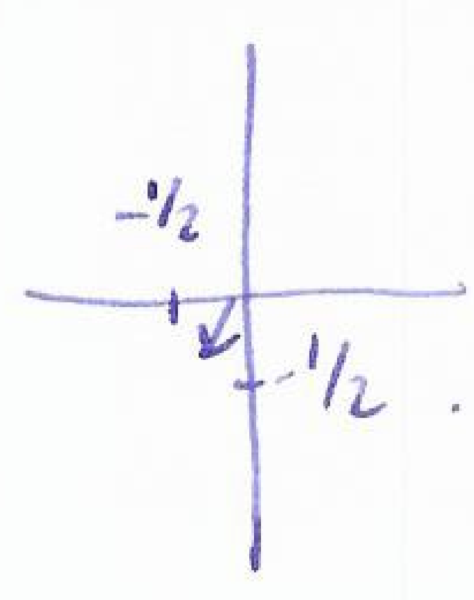
<u>Example</u> $\underline{v} = \langle 1,1\rangle$



$4\underline{v} =$

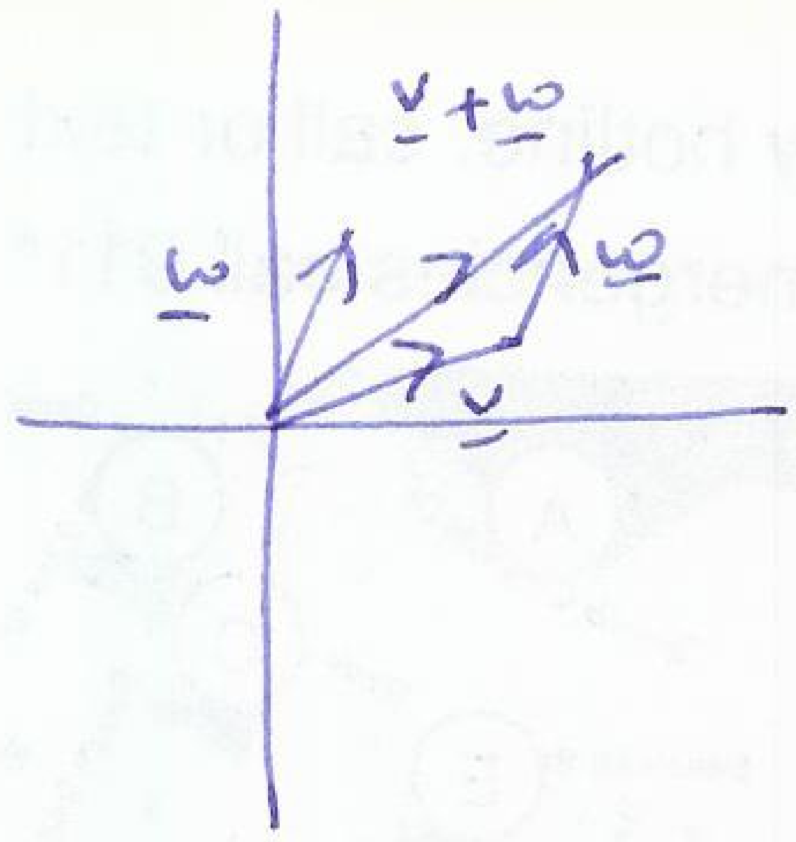


$-\frac{1}{2}\underline{v} =$

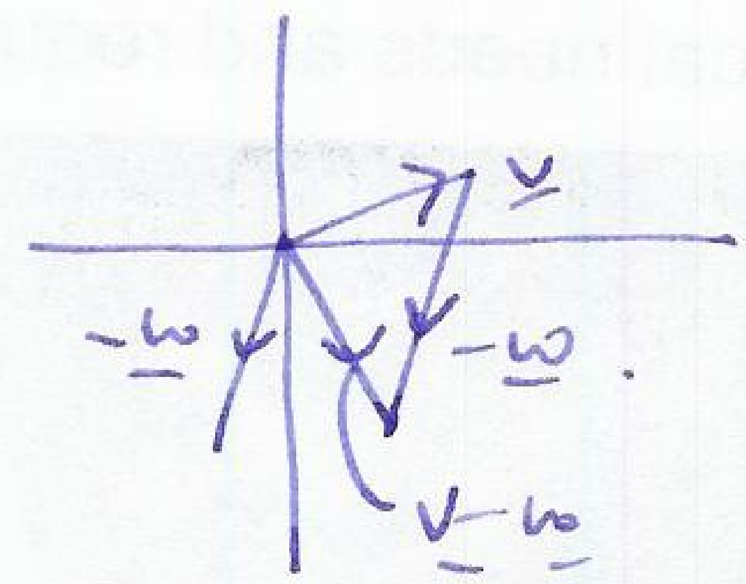


<u>observation</u> $\underline{v}$ is parallel to $\underline{w}$ iff $\underline{v} = \lambda\underline{w}$ for some λ

<u>vector addition</u> • $\underline{v},\underline{w}$ vectors. translate $\underline{w}$ so that beginning of $\underline{w}$ at end of $\underline{v}$, then $\underline{v}+\underline{w}$ is vector from beginning of $\underline{v}$ to end of $\underline{w}$.



$\underline{v}-\underline{w} = \underline{v} + (-1)\underline{w}$.



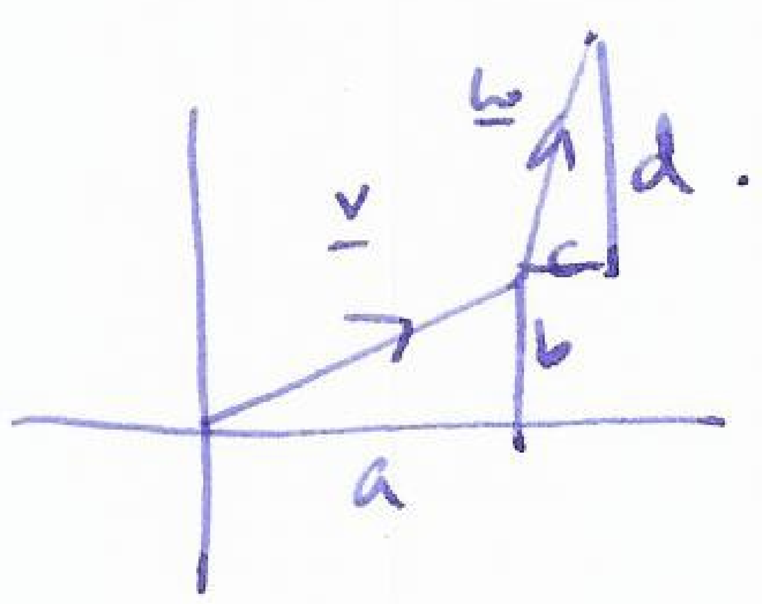
vector operations in components.

if $\underline{v} = \langle a,b \rangle$ $\underline{w} = \langle c,d \rangle$

then
- $\lambda \underline{v} = \langle \lambda a, \lambda b \rangle$.
- $\underline{v}+\underline{w} = \langle a,b \rangle + \langle c,d \rangle = \langle a+c, b+d \rangle$
- $\underline{v}-\underline{w} = \langle a,b \rangle - \langle c,d \rangle = \langle a-c, b-d \rangle$
- $\underline{v}+\underline{0} = \langle a,b \rangle + \langle 0,0 \rangle = \langle a,b \rangle = \underline{0}+\underline{v}$

important $\underline{v}-\underline{v} = \underline{0} = \langle 0,0 \rangle$ zero vector not zero number.

check:



Useful properties

$\underline{u},\underline{v},\underline{w}$ vectors, λ scalar

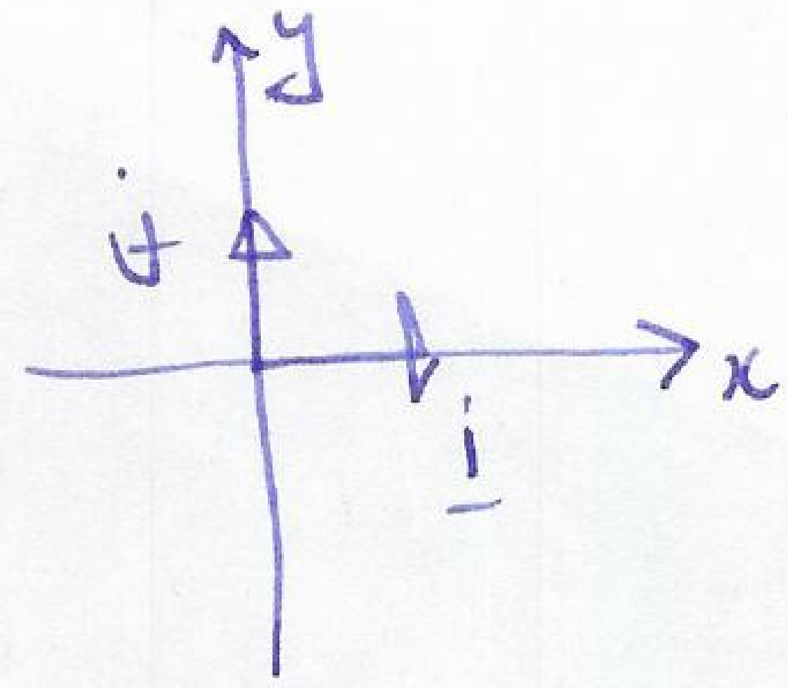
- $\underline{u}+\underline{v} = \underline{v}+\underline{w}$ (commutative)
- $\underline{u}+(\underline{v}+\underline{w}) = (\underline{u}+\underline{v})+\underline{w}$ (associative)
- $\lambda(\underline{v}+\underline{w}) = \lambda\underline{v}+\lambda\underline{w}$ (distributive)
- length $\|\lambda \underline{v}\| = |\lambda| \, \|\underline{v}\|$

important $\lambda + \underline{v}$ does not make sense!

unit vectors : A vector of length 1 is called a unit vector.

if $\underline{v}$ is a (non-zero) vector, then $\hat{v} = \hat{e}_v = \frac{1}{\|\underline{v}\|}\underline{v}$ is a unit vector in the direction of $\underline{v}$. Check $\left\|\frac{1}{\|\underline{v}\|}\underline{v}\right\| = \left|\frac{1}{\|\underline{v}\|}\right| \|\underline{v}\| = \frac{\|\underline{v}\|}{\|\underline{v}\|} = 1$.

special vectors



$\underline{i}$ unit vector in x-direction

$\underline{j}$ unit vector in y-direction

so $\underline{i} = \langle 1,0 \rangle$ $\underline{j} = \langle 0,1 \rangle$

i,j called standard basis vectors.

linear combinations

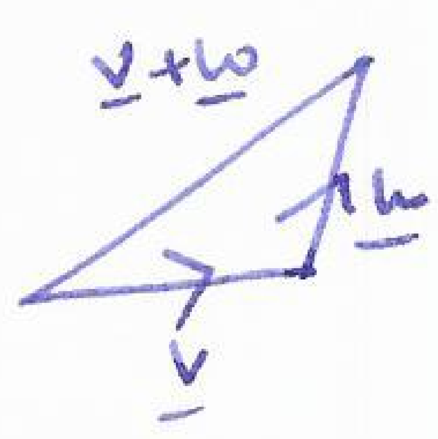
$\underline{v}, \underline{w}$ vector r, s scalars then

$r\underline{v} + s\underline{w}$ is a linear combination of $\underline{v}$ and $\underline{w}$

Every vector can be written as a linear combination of $\underline{i}$ and $\underline{j}$.

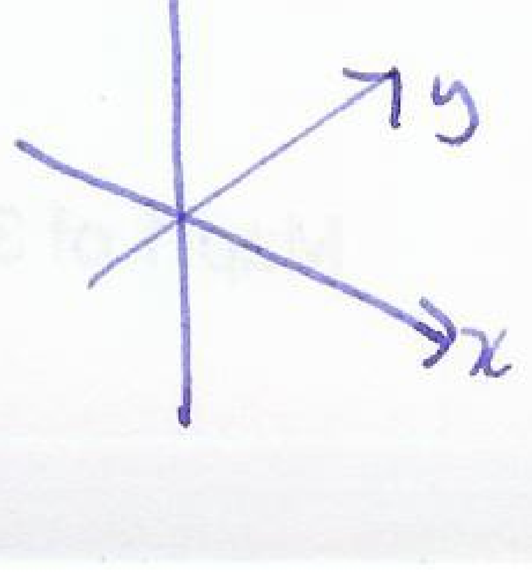
$\underline{v} = \langle a,b \rangle = a\langle 1,0 \rangle + b\langle 0,1 \rangle = a\underline{i} + b\underline{j}$.

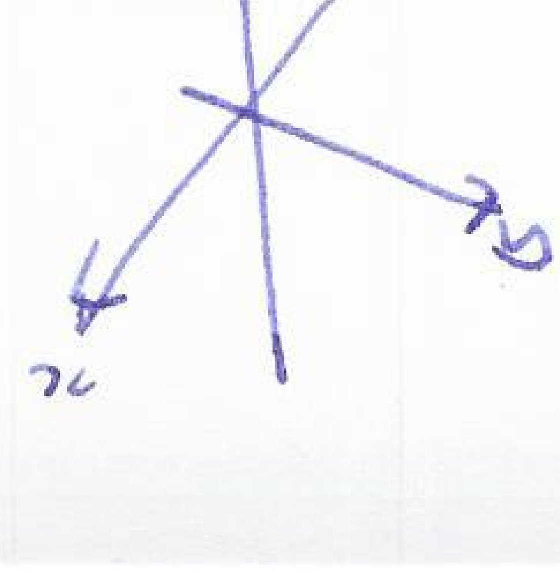
triangle inequality



$\|\underline{v}+\underline{w}\| \leq \|\underline{v}\| + \|\underline{w}\|$

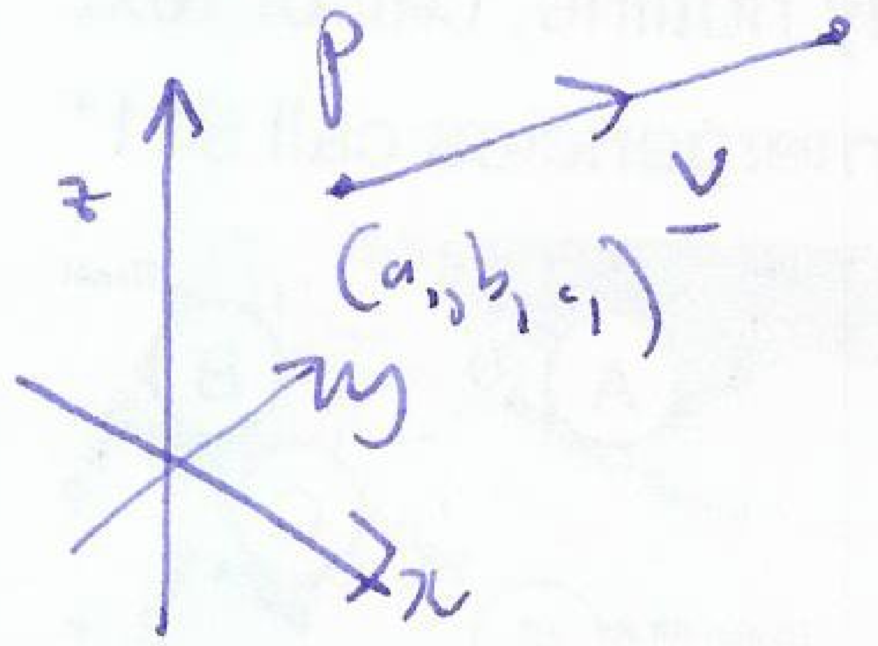
§12.2 Vectors in 3d





right hand rule

A vector $\underline{v}$ has length and direction.

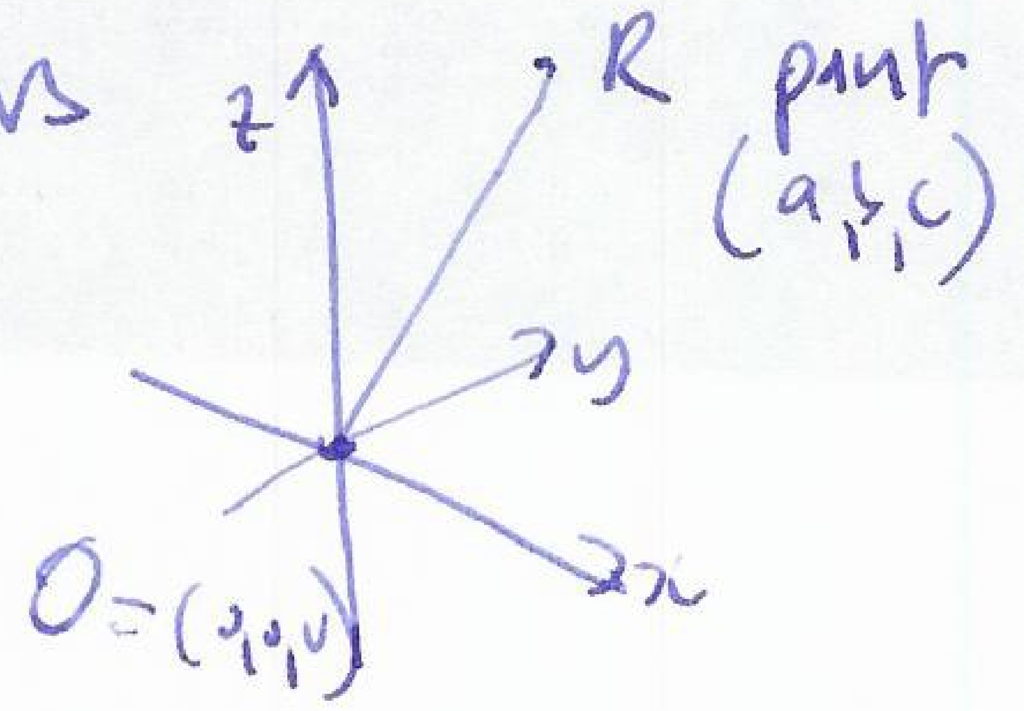


- $\underline{v}$ is determined by its initial and final points.

- length $|\overrightarrow{PQ}| = \sqrt{(a_2-a_1)^2 + (b_2-b_1)^2 + (c_2-c_1)^2}$

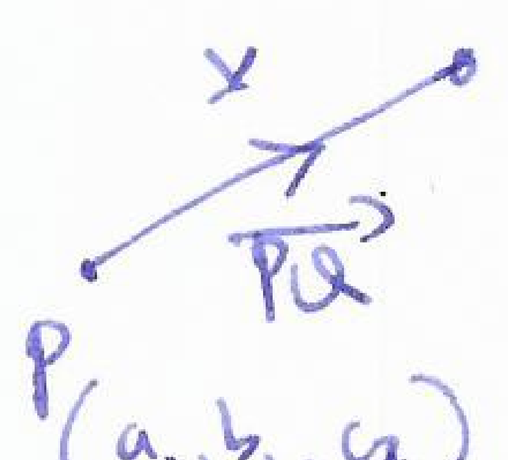
- translation: move $\underline{v}$ without changing length or direction
- equal: $\underline{v}, \underline{w}$ are equal if they are translates of each other, i.e. have same length and direction.

- position vectors



$\overrightarrow{OR} = \langle a, b, c \rangle$.
position vector.

<u>components</u>:



components are

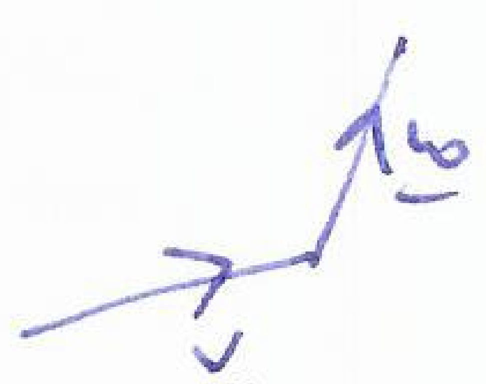
x-component $a_2 - a_1$

y-component $b_2 - b_1$

z-component $c_2 - c_1$

notation $\overrightarrow{PQ} = \langle a_2 - a_1, b_2 - b_1, c_2 - c_1 \rangle$

- <u>vector addition</u>



$\underline{v} = \langle v_1, v_2, v_3 \rangle$

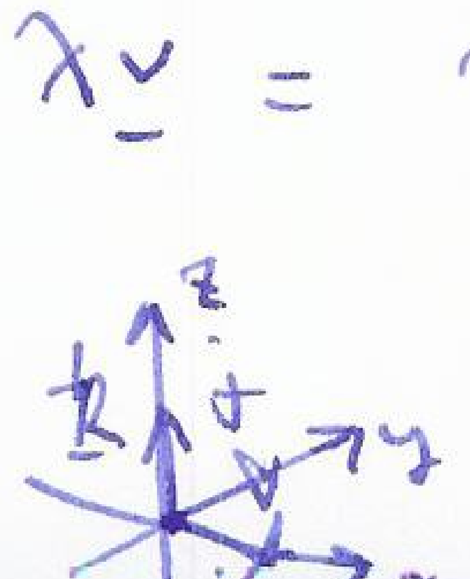
$\underline{w} = \langle w_1, w_2, w_3 \rangle$

$\underline{v} + \underline{w} = \langle v_1 + w_1, v_2 + w_2, v_3 + w_3 \rangle$

- <u>scalar multiplication</u>

$\lambda \underline{v} = \lambda \langle v_1, v_2, v_3 \rangle = \langle \lambda v_1, \lambda v_2, \lambda v_3 \rangle$

- standard basis vectors



$i = \langle 1, 0, 0 \rangle$

$j = \langle 0, 1, 0 \rangle$

$k = \langle 0, 0, 1 \rangle$

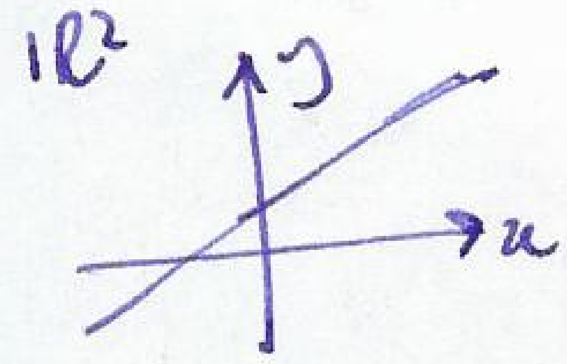
every vector is a <u>linear combination</u> of the standard basis vectors.

$\underline{v} = \langle a, b, c \rangle = a\underline{i} + b\underline{j} + c\underline{k} = a\langle 1,0,0\rangle + b\langle 0,1,0\rangle + c\langle 0,0,1\rangle$

$= \langle a,0,0\rangle + \langle 0,b,0\rangle + \langle 0,0,c\rangle$

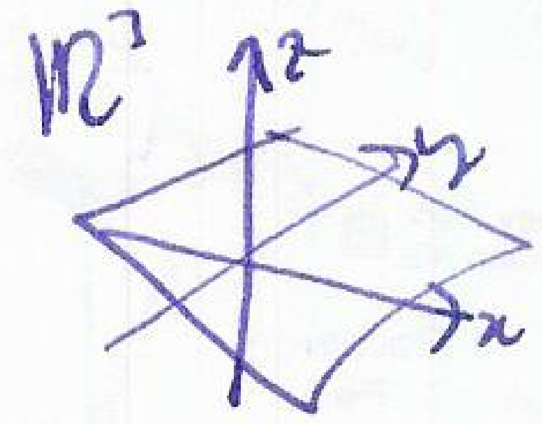
$= \langle a,b,c\rangle.$

<u>Equations of lines in $\mathbb{R}^3$</u>

recall. 

~~eq~~ $y = mx + b$

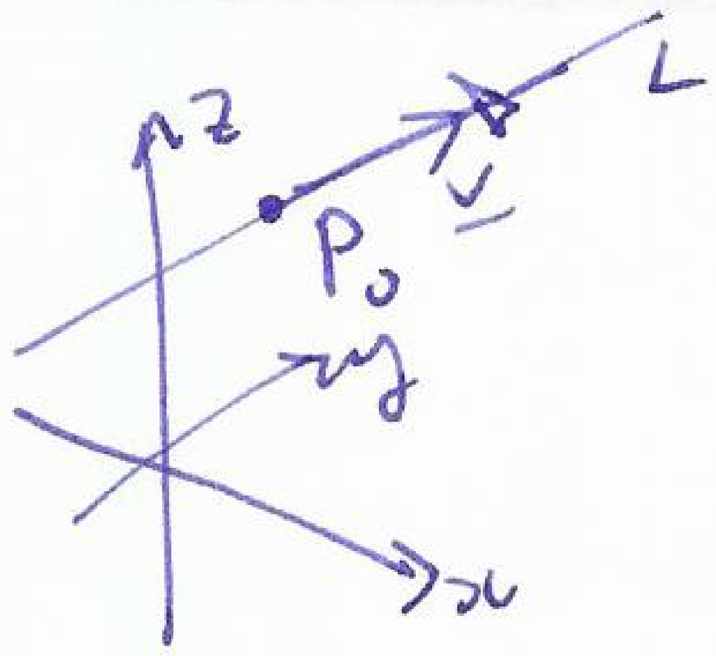
line $\Leftrightarrow$ single linear equation



$ax + by + cz = d$

single linear equation is a plane!

for lines in $\mathbb{R}^3$ we can use a <u>parametric equation</u> $r(t)$



a line L is determined by: a point P_0 on L

a direction $\underline{v}$

any point on L can then be written as $\underline{r}(t) = \overrightarrow{OP_0} + t\underline{v}$

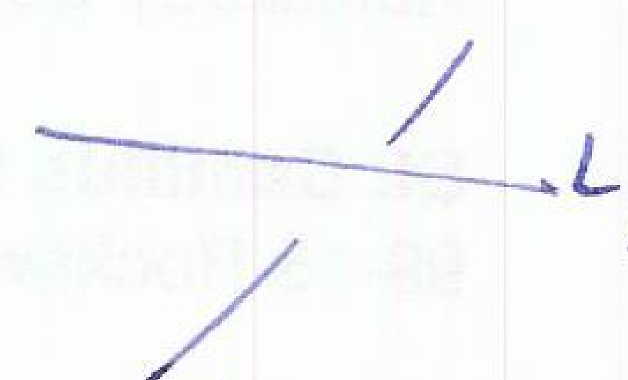
t is called the <u>parameter</u> if $\overrightarrow{OP_0} = \langle a, b, c\rangle$

$\underline{v} = \langle v_1, v_2, v_3\rangle$

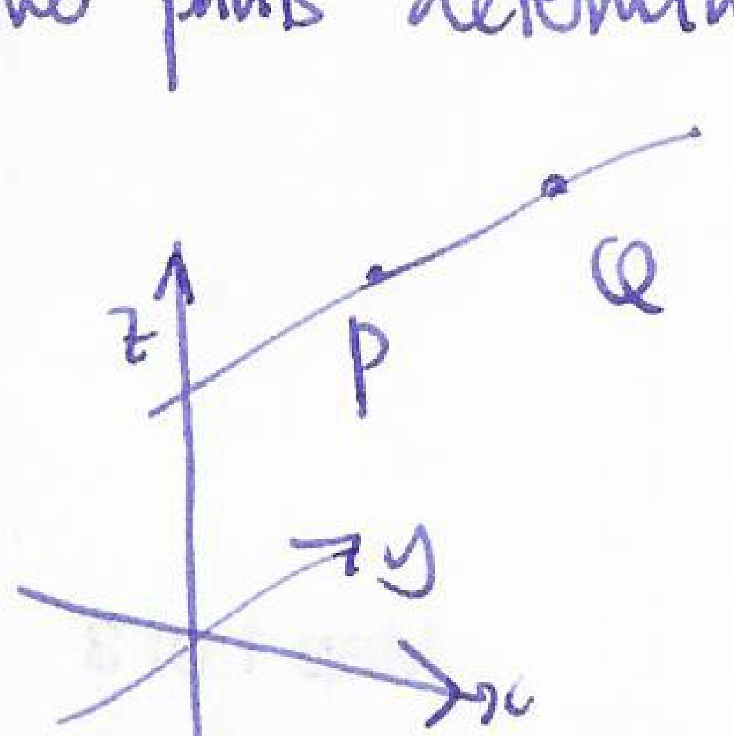
then $\underline{r}(t) = \langle a,b,c\rangle + t\langle v_1, v_2, v_3\rangle = \langle a + tv_1, b + tv_2, c + tv_3\rangle.$

$\mathbb{R}^2$: any two lines either intersect or are parallel.

$\mathbb{R}^3$: lines may intersect, or be parallel, or <u>neither</u> (skew)



Two points determine a line



$P = (p_1, p_2, p_3)$

$Q = (q_1, q_2, q_3)$

then $\underline{v} = \overrightarrow{PQ} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle.$

so $\underline{r}(t) = \langle p_1 + t(q_1 - p_1), p_2 + t(q_2 - p_2), p_3 + t(q_3 - p_3)\rangle$