Math 233 Calculus 3 Spring 13 Midterm 3b

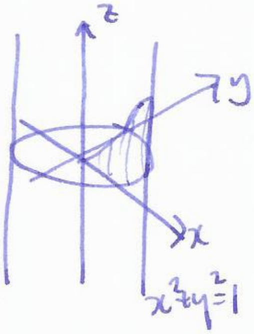
Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 3	
Overall	

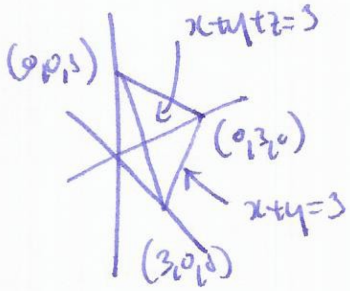
(1) (10 points) Write down limits for the integral over the region in the positive octant below the surface $z = 2xy$ and inside the cylinder $x^2 + y^2 = 1$.



$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$\int_0^1 \int_0^{\frac{\pi}{2}} \int_0^{2r^2\cos\theta\sin\theta} r\,dz\,d\theta\,dr$$

(2) (10 points) Use a triple integral to find the volume of the tetrahedron with vertices $(0,0,0)$, $(3,0,0)$, $(0,3,0)$ and $(0,0,3)$.



$$\int_0^3 \int_0^{3-x} \int_0^{3-x-y} dz\,dy\,dx$$

$$\left[z\right]_0^{3-x-y} = 3-x-y$$

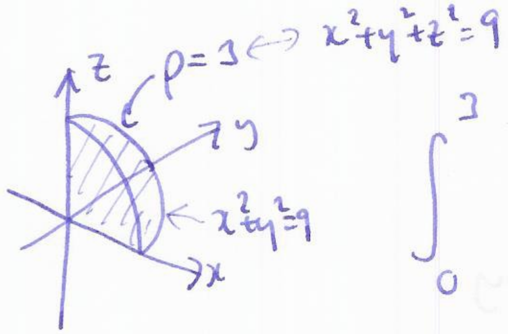
$$\int_0^{3-x} 3-x-y\,dy = \left[(3-x)y - \frac{1}{2}y^2\right]_0^{3-x} = \frac{1}{2}(3-x)^2$$

$$\int_0^3 \frac{9}{2} - 3x + \frac{1}{2}x^2\,dx = \left[\frac{9}{2}x - \frac{3}{2}x^2 + \frac{1}{6}x^3\right]_0^3$$

$$= \frac{27}{2} - \frac{27}{2} + \frac{27}{6} = \frac{9}{2}$$

(3) (10 points) Draw a picture of the region described by the limits of the following integral, and write down limits for the region interms of cartesian coordinates.

$$\int_0^3 \int_0^{\pi/2} \int_0^{\pi/2} d\theta \, d\phi \, d\rho$$



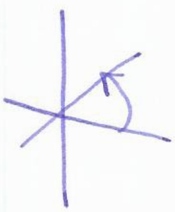
$$\int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} dz \, dy \, dx$$

(4) (10 points) Use the change of variable $T(u, v) = (u - uv, uv)$ to evaluate

$$\int\int_D \sqrt{x+y}\, dx\, dy,$$

where D is the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$.

(5) (10 points) Let $f(x, y, z) = -x + y + z$. Evaluate $\int_C f \, ds$, where C is the portion of the unit circle in the xy-plane which lies in the positive octant, oriented counter-clockwise.



$\underline{c}(\theta) = (\cos\theta, \sin\theta, 0) \quad 0 \le \theta \le \frac{\pi}{2}$

$\underline{c}'(\theta) = (-\sin\theta, \cos\theta, 0)$

$\|c'(\theta)\| = 1$

$\int_0^{\pi/2} -\cos\theta + \sin\theta + 0 \, d\theta = \left[-\sin\theta - \cos\theta\right]_0^{\pi/2} = (-1-0) - (0-1) = 0$

(6) (10 points) Show that the vector field $\mathbf{F} = \langle -z, x, y\rangle$ is not conservative, and evaluate $\int_C \mathbf{F}\, d\mathbf{s}$, where C is the straight line from the origin to the point $(1, 2, 2)$.

$$\operatorname{curl}(\underline{F}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z & x & y \end{vmatrix} = \langle 1, 1, 1\rangle \neq \underline{0} \Rightarrow \text{not conservative}$$

$$\underline{c}(t) = (t, 2t, 2t) \quad 0 \leq t \leq 1$$

$$\underline{c}'(t) = (1, 2, 2)$$

$$\int_0^1 (-2t, t, 2t)\cdot(1,2,2)\, dt = \int_0^1 -2t + 2t + 4t\, dt = \int_0^1 4t\, dt$$

$$= \left[2t^2\right]_0^1 = 2$$

(7) (10 points) Show that the vector field $\mathbf{F} = \langle -y, z-x, y \rangle$ is conservative, and find $\int_C \mathbf{F}\, d\mathbf{s}$, where C is the shortest path on the unit cube from $(0,0,0)$ to $(1,1,1)$.

$F_1: \int -y\,dx = -xy + c_1(y,z)$

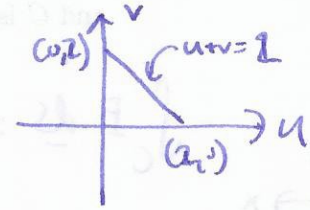
$F_2: \int z-x\,dy = zy - xy + c_2(x,z)$

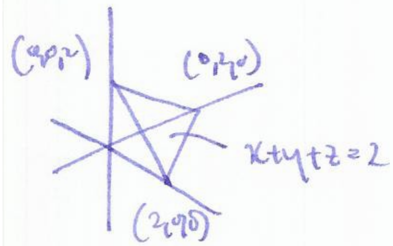
$F_3: \int y\,dz = zy + c_3(x,y)$

$f(x,y,z) = -xy + zy + c$

$$\int_C \mathbf{F} \cdot d\mathbf{s} = f(1,1,1) - f(0,0,0) = -1+1 = 0$$

(8) (10 points) Integrate the vector field $\mathbf{F} = \langle x, y, z\rangle$ over the triangle with vertices $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$.





$$T(u,v) = (u, v, 2-u-v)$$

$$\frac{\partial T}{\partial u} = (1,0,-1)$$

$$\frac{\partial T}{\partial v} = (0,1,-1)$$

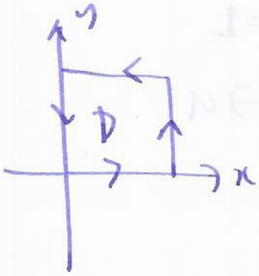
$$\underline{n} = \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1\rangle$$

$$\int_0^2 \int_0^{2-u} (u, v, 2-u-v)\cdot(1,1,1)\, dv\, du = \int_0^2 \int_0^{2-u} \cancel{u}+\cancel{v}+2-\cancel{u}-\cancel{v}\, dv\, du$$

$$= \int_0^2 \underbrace{\int_0^{2-u} 2\, dv}_{[2v]_0^{2-u} = 4-2u}\, du$$

$$\int_0^2 4-2u\, du = \left[4u-u^2\right]_0^2 = 8-4 = 4$$

(9) (10 points) Use Green's Theorem to evaluate $\int_C \mathbf{F}\, d\mathbf{s}$, where $\mathbf{F} = \langle e^{2y}, e^{-2x} \rangle$, and C is the boundary of the unit square, oriented counter-clockwise.



$$\int_C \underline{F} \cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

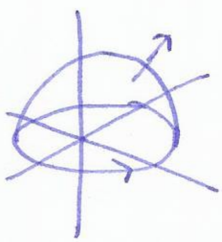
$$\int_0^1 \underbrace{\int_0^1 -2e^{-2x} - 2e^{2y}\, dy}\, dx$$

$$\underbrace{\left[-2e^{-2x} \cdot y - e^{2y}\right]_0^1 = -2e^{-2x} - e^2 - (0 - 1)}$$

$$\int_0^1 -2e^{-2x} - e^2 + 1\, dx = \left[\frac{4}{2} e^{-2x} + (1 - e^2)x\right]_0^1$$

$$= e^{-2} + (1 - e^2) - 1 = e^{-2} - e^2$$

(10) (10 points) Use Stokes' Theorem to evaluate the integral of curl($\mathbf{F}$) over the unit hemisphere $x^2 + y^2 + z^2 = 4$, with $z \geq 0$, where $\mathbf{F} = \langle -y, x, z \rangle$.



$$\iint_S \text{curl}(F) \cdot d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{s}$$

$\underline{c}(\theta) = (2\cos\theta, 2\sin\theta, 0) \quad 0 \leq \theta \leq 2\pi$

$\underline{c}'(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

$$\int_0^{2\pi} \langle -2\sin\theta, 2\cos\theta, 0 \rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0 \rangle \, d\theta$$

$$= \int_0^{2\pi} 4 \, d\theta = \left[4\theta\right]_0^{2\pi} = 8\pi$$