Math 233 Calculus 3 Spring 13 Midterm 3a

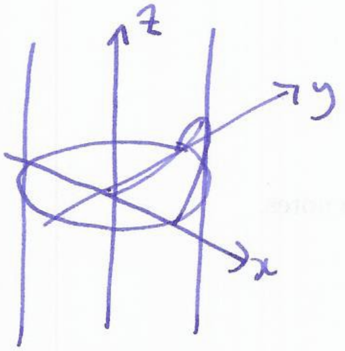
Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Midterm 3	
Overall	

(1) (10 points) Write down limits for the integral over the region in the positive octant below the surface $z = xy$ and inside the cylinder $x^2 + y^2 = 4$.



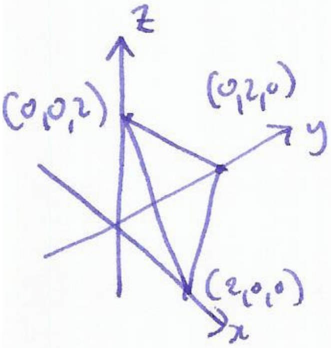
$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\int_0^2 \int_0^{\pi/2} \int_0^{r^2\cos\theta\sin\theta} r \, dz \, d\theta \, dr$$

(2) (10 points) Use a triple integral to find the volume of the tetrahedron with vertices $(0,0,0)$, $(2,0,0)$, $(0,2,0)$ and $(0,0,2)$.



$$\int_0^2 \int_0^{2-x} \underbrace{\int_0^{2-x-y} 1 \, dz}_{} \, dy \, dx$$

$$\underbrace{\Big[z \Big]_0^{2-x-y} = 2-x-y}_{}$$

$$\int_0^{2-x} 2-x-y \, dy = \Big[(2-x)y - \tfrac{1}{2}y^2 \Big]_0^{2-x} = \tfrac{1}{2}(2-x)^2$$

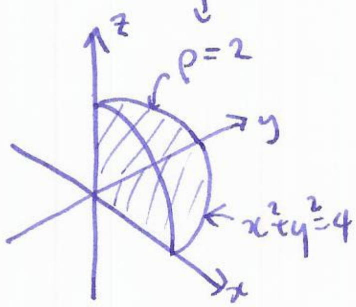
$$\int_0^2 2 - 2x + \tfrac{1}{2}x^2 \, dx = \Big[2x - \tfrac{2}{2}x^2 + \tfrac{1}{6}x^3 \Big]_0^2 = 4 - 4 + \tfrac{4}{3} = \tfrac{4}{3}$$

(3) (10 points) Draw a picture of the region described by the limits of the following integral, and write down limits for the region interms of cartesian coordinates.

$$\int_0^{\pi/2}\int_0^{\pi/2}\int_0^{2} d\rho\, d\phi\, d\theta$$

$x^2+y^2+z^2=4$

$\Updownarrow$

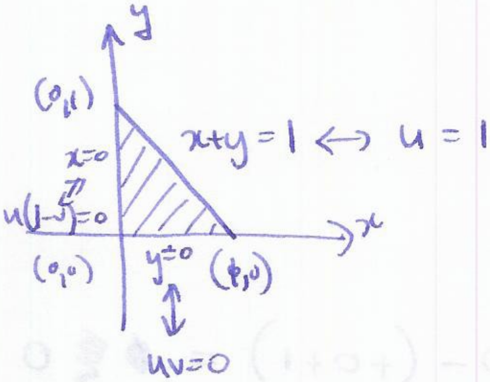


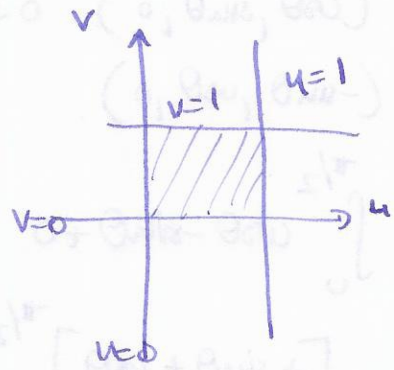
$$\int_0^{2}\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} dz\, dy\, dx$$

(4) (10 points) Use the change of variable $T(u, v) = (u - uv, uv)$ to evaluate

$$\iint_D \sqrt{x+y}\, dx\, dy,$$

where D is the triangle with vertices $(0,0)$, $(0,1)$ and $(1,0)$.



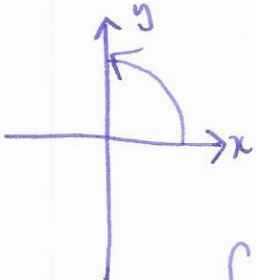


$$\int_0^1 \int_0^1 \sqrt{u}\, J\, du\, dv \qquad J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

$$= \int_0^1 \underbrace{\int_0^1 u^{3/2}\, du}\, dv = \left[\frac{2}{5}u^{5/2}\right]_0^1 = \frac{2}{5}.$$

$$\int_0^1 \frac{2}{5}\, dv = \left[\frac{2}{5}v\right]_0^1 = \frac{2}{5}$$

(5) (10 points) Let $f(x, y, z) = x - y + z$. Evaluate $\int_C f\, ds$, where C is the portion of the unit circle in the xy-plane which lies in the positive octant, oriented counter-clockwise.



$c(\theta) = (\cos\theta, \sin\theta, 0) \quad 0 \leq \theta \leq \frac{\pi}{2}$

$c'(\theta) = (-\sin\theta, \cos\theta, 0). \quad \|c'(\theta)\| = 1$

$\int_C f\, ds = \int_0^{\pi/2} \cos\theta - \sin\theta + 0 \;\; 1\, d\theta$

$= \left[+ \sin\theta + \cos\theta \right]_0^{\pi/2} = +1+0-(+0+1) = 0$

(6) (10 points) Show that the vector field $\mathbf{F} = \langle z, x, -y \rangle$ is not conservative, and evaluate $\int_C \mathbf{F}\, d\mathbf{s}$, where C is the straight line from the origin to the point $(1, 2, 2)$.

$$\text{curl}(\underline{F}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & -y \end{vmatrix} = \langle -1, 1, 1 \rangle \neq \underline{0} \rightarrow \text{not conservative.}$$

$\underline{c}(t) = (t, 2t, 2t) \quad 0 \leq t \leq 1$

$\underline{c}'(t) = (1, 2, 2)$

$$\int_0^1 \langle 2t, t, -2t \rangle \cdot \langle 1, 2, 2 \rangle \, dt = \int_0^1 \underbrace{2t + 2t - 4t}_{0} \, dt = 0$$

(7) (10 points) Show that the vector field $\mathbf{F} = \langle y, x - z, -y \rangle$ is conservative, and find $\int_C \mathbf{F}\, d\mathbf{s}$, where C is the shortest path on the unit cube from $(0, 0, 0)$ to $(1, 1, 1)$.

F_1: $\int y\, dx = xy + c_1(y,z)$

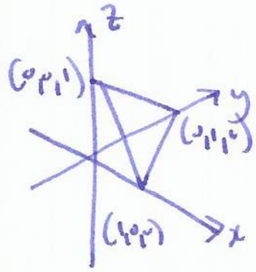
F_2: $\int x - z\, dy = xy - yz + c_2(x,z)$

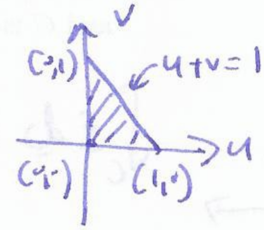
F_3: $\int -y\, dz = -yz + c_3(x,y)$

$f(x,y,z) = xy - yz + C$

$$\int_C F \cdot ds = f(\text{end}) - f(\text{start}) = f(1,1,1) - f(0,0,0) = 1 - 1 - 0 = 0$$

(8) (10 points) Integrate the vector field $\mathbf{F} = \langle 2x, 2y, 2z \rangle$ over the triangle with vertices $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.





$$T(u,v) = (u, v, 1-u-v)$$

$$\frac{\partial T}{\partial u} = \langle 1, 0, -1 \rangle$$

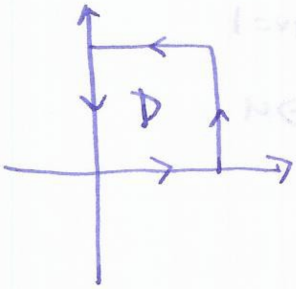
$$\frac{\partial T}{\partial v} = \langle 0, 1, -1 \rangle$$

$$\underline{n} = \frac{\partial T}{\partial u} \times \frac{\partial T}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \langle 1, 1, 1 \rangle$$

$$\int_0^1 \int_0^{1-u} \langle 2u, 2v, 2-2u-2v \rangle . \langle 1,1,1 \rangle \, dv\, du = \int_0^1 \underbrace{\int_0^{1-u} \cancel{2u} + \cancel{2v} + 2 - \cancel{2u} - \cancel{2v} \, dv}_{[2v]_0^{1-u} = 2-2u} du$$

$$\int_0^1 2 - 2u \, du = \left[2u - u^2 \right]_0^1 = 2 - 1 = 1$$

(9) (10 points) Use Green's Theorem to evaluate $\int_C \mathbf{F}\, d\mathbf{s}$, where $\mathbf{F} = \langle e^{-2y}, e^{2x} \rangle$, and C is the boundary of the unit square, oriented counter-clockwise.



$$\int_C \underline{F}\cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

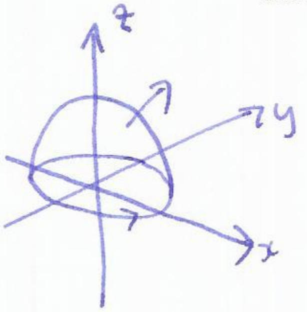
$$\int_0^1 \int_0^1 2e^{2x} + 2e^{-2y}\, dy\, dx$$

$$\left[2e^{2x}\cdot y + e^{-2y}\right]_0^1 = 2e^{2x} + e^{-2} - (0+1)$$

$$\int_0^1 2e^{2x} + e^{-2} - 1\, dx = \left[e^{2x} + (e^{-2}-1)x\right]_0^1 = e^{2} + e^{-2} - 1 - 1 = \text{[illegible]}$$

$$= e^{2} + e^{-2} - 2$$

(10) (10 points) Use Stokes' Theorem to evaluate the integral of curl($\mathbf{F}$) over the unit hemisphere $x^2 + y^2 + z^2 = 1$, with $z \geq 0$, where $\mathbf{F} = \langle -2y, 2x, 2z \rangle$.



$$\iint_S \text{curl}(\underline{F}) \, d\underline{S} = \int_{\partial S} \underline{F} \cdot d\underline{s}$$

$$\underline{c}(\theta) = (\cos\theta, \sin\theta, 0)$$

$$\underline{c}'(\theta) = (-\sin\theta, \cos\theta, 0)$$

$$\int_0^{2\pi} \langle -2\sin\theta, 2\cos\theta, 0 \rangle \cdot \langle -\sin\theta, \cos\theta, 0 \rangle \, d\theta$$

$$\int_0^{2\pi} 2\sin^2\theta + 2\cos^2\theta \, d\theta = \int_0^{2\pi} 2 \, d\theta = 4\pi$$