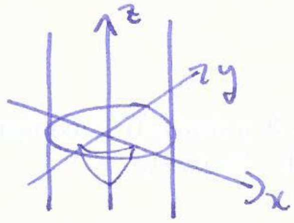
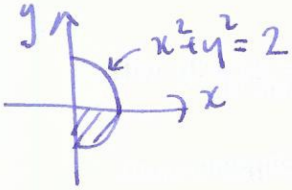
Sample midterm 3 solutions

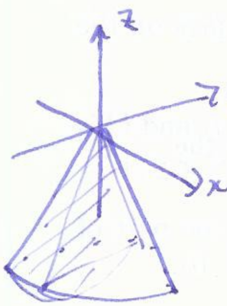
Q1 a)



$$\int_0^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{0}\int_{-\sqrt{4-2x^2-2y^2}}^{0} f(x,y,z)\,dz\,dy\,dx$$



b)



$x \leq 0$

$$\int_{2.82}^{\pi}\int_{\frac{1}{2}\pi}^{2\pi}\int_0^5 f(\rho,\theta,\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$x = \rho\cos\theta\sin\phi$

$y = \rho\sin\theta\sin\phi$.

$z = \rho\cos\phi$

$z^2 = +3x^2+3y^2$. with $z \leq 0$

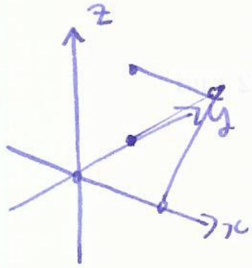
$\updownarrow$

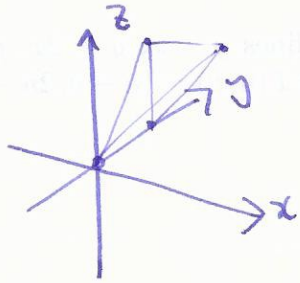
$\rho^2\cos^2\phi = +3\left(\rho^2\cos^2\theta\sin^2\phi + \rho^2\sin^2\theta\sin^2\phi\right)$

$\cos^2\phi = +3\sin^2\phi$.

$\tan^2\phi = \frac{1}{3}$ $\qquad \phi \approx 2.82$.

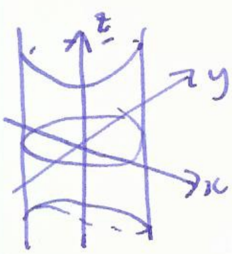
c)





$$\int_0^1\int_0^y\int_0^z f(x,y,z)\,dx\,dz\,dy \ \cancel{dz\,dy}$$

Q2



$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$$\int_0^{2\pi}\int_0^2\int_{-2r^2(\cos\theta-\sin\theta)^2}^{2r^2(\cos\theta+\sin\theta)^2} 1\,r\,dz\,dr\,d\theta$$

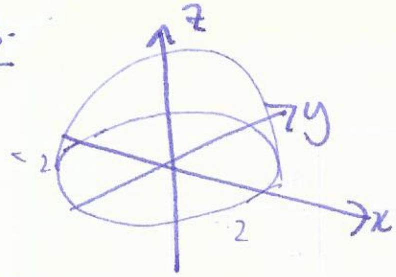
$2r^2\left(\cos^2\theta + 2\sin\theta\cos\theta + \sin^2\theta\right)$

$+\cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta$

$$\int_0^2 r^3\left(\overset{2}{\cancel{8\sin\theta\cos\theta}}\right)dr = \left[\frac{1}{2}r^4\right]_0^2 = \overset{8}{\cancel{4(\ )}}$$

$$\int_0^{2\pi} 8\,d\theta = 16\pi.$$

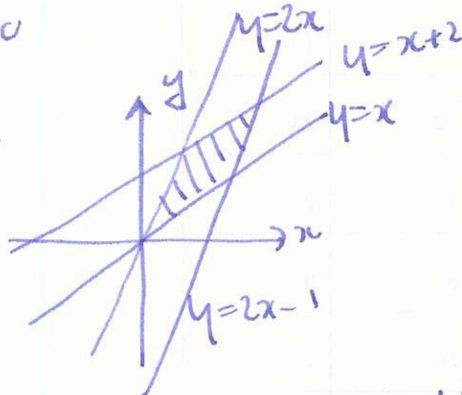
Q3.



$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^{2} e^{-\rho^3}\rho^2\sin\phi \, d\rho\, d\phi\, d\theta \quad (*)$$

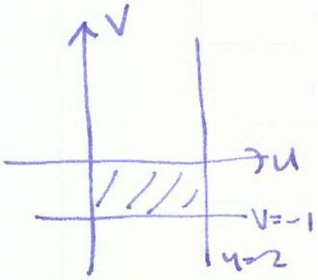
$$\left.\begin{aligned}
\int_0^2 \rho^2 e^{-\rho^3}\,d\rho &= \left[-\tfrac{1}{3}e^{-\rho^3}\right]_0^2 = -\tfrac{1}{3}e^{-8}+\tfrac{1}{3}\\
\int_0^{\pi/2}\sin\phi\,d\phi &= \left[-\cos\phi\right]_0^{\pi/2} = -0+1\\
\int_0^{2\pi} 1\,d\theta &= 2\pi
\end{aligned}\right\} \Rightarrow (*) = \frac{2\pi}{3}\left(1-e^{-8}\right).$$

Q4



$x = u-v$
$y = 2u-v$

$y=x \Leftrightarrow 2u-v = u-v \Leftrightarrow u=0$
$y=2x \Leftrightarrow 2u-v = 2u-2v \Leftrightarrow v=0.$
$y=x+2 \Leftrightarrow 2u-v = u-v+2 \Leftrightarrow u=2$
$y=2x-1 \Leftrightarrow 2u-v = 2u-2v-1 \Leftrightarrow v=-1.$



$$J = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix}\right| = \left|\begin{vmatrix} 1 & -1 \\ 2 & -1\end{vmatrix}\right| = (-1+2) = 1.$$

$$\int_{-1}^{0}\int_0^2 \underbrace{(2u-v-u+v)^2}_{u^2}\, 1 \, du\, dv$$

$$\int_0^2 u^2\,du = \left[\tfrac{1}{3}u^3\right]_0^2 = \frac{8}{3}. \qquad \int_{-1}^{0}\frac{8}{3}\,dv = -\frac{8}{3}.$$

Q5

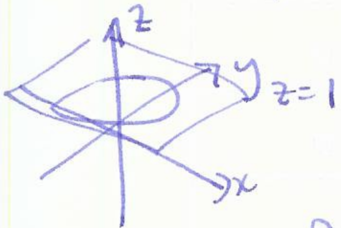
$\underline{c}(t) = t(4,6,6)+(1-t)(-1,3,2) = (-1+5t,\ 3+3t,\ 2+4t)$

$\underline{c}'(t) = (4,6,6)-(-1,3,2) = (5,3,4) \qquad \|c'(t)\| = \sqrt{50}.$

$$\int_0^1\left(e^{-1+5t} + \underbrace{(3+3t)(2+4t)}_{12t^2+18t+6}\right)\sqrt{50}\,dt = \sqrt{50}\left[\tfrac{1}{5}e^{-1+5t} + 4t^3+9t^2+6t\right]_0^1 = \left(\tfrac{1}{5}e^{4}+19\right)\sqrt{50}$$

Q6 $\underline{\nabla} \times \underline{F} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & -z \end{vmatrix} = \langle 0, 0, 1-2y \rangle \neq \underline{0} \Rightarrow$ not conservative.

$\underline{F} = \langle y^2, x, -z \rangle$



$\underline{c}(t) = \langle 4\cos\theta, 4\sin\theta, 1 \rangle$

$\underline{c}'(t) = \langle -4\sin\theta, 4\cos\theta, 0 \rangle.$

$$\int_C \underline{F} \cdot \underline{ds} = \int_0^{2\pi} \langle 16\sin^2\theta, 4\cos\theta, -1 \rangle \cancel{\langle 0, 0, 1-8\sin\theta \rangle} \cdot \langle -4\sin\theta, 4\cos\theta, 0 \rangle \, d\theta$$

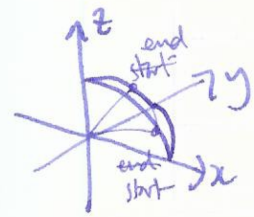
Q7 want $\underline{F} = \nabla f$

$\frac{\partial f}{\partial x} = z e^{xz} \qquad f = e^{xz} + f_1(y, z).$

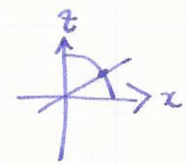
$\frac{\partial f}{\partial y} = -z\sin(yz) \qquad f = \cos(yz) + f_2(x, z).$

$\frac{\partial f}{\partial z} = x e^{xz} - y\sin(yz) \qquad f = e^{xz} + \cos(yz) + f_3(x, y).$

$f(x, y, z) = e^{xz} + \cos(yz) + c$ so $\int_C \underline{F} \cdot \underline{ds} = f(\text{end}) - f(\text{start}).$



$\left.\begin{matrix} z = 2x + 3y \\ x^2 + y^2 + z^2 = 36^2 \end{matrix}\right\}$ $y = 0$: $\begin{matrix} z = 2x \\ x^2 + z^2 = 36^2 \end{matrix}$



$5x^2 = 36^2$
$x = \frac{36}{\sqrt{5}}$

$\left(\frac{36}{\sqrt{5}}, 0, \frac{72}{\sqrt{5}} \right).$

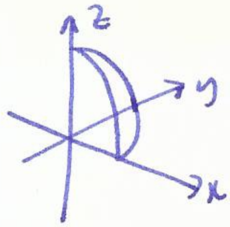
$x = 0$: $\begin{matrix} z = 3y \\ y^2 + z^2 = 36^2 \end{matrix}$ $\qquad 10y^2 = 36^2 \qquad y = \frac{36}{\sqrt{10}}.$

$\left(0, \frac{36}{\sqrt{10}}, \frac{72}{\sqrt{10}} \right).$

$f(\text{end}) - f(\text{start}) = f\left(0, \frac{36}{\sqrt{10}}, \frac{72}{\sqrt{10}}\right) - f\left(\frac{36}{\sqrt{5}}, 0, \frac{72}{\sqrt{5}}\right).$

$= 1 + \cos\left(\frac{36 \times 72}{10}\right) - e^{\frac{36 \times 72}{5}} \cancel{+}\, 1.$

Q8



$$z = 16 - x^2 - y^2$$

parametrize: $\Phi(r,\theta) = (r\cos\theta, r\sin\theta, 16 - r^2)$

$$\frac{\partial \Phi}{\partial r} = \langle \cos\theta, \sin\theta, -2r \rangle$$

$$\frac{\partial \Phi}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\underline{n} = \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} = \langle 2r^2\cos\theta, 2r^2\sin\theta, r(\cos^2\theta + \sin^2\theta) \rangle$$

$$\int_0^4 \int_0^{\pi/2} 1\sqrt{4r^4 + r^2}\, d\theta\, dr = \frac{\pi}{2}\int_0^4 r\sqrt{4r^2+1}\, dr = \left[\frac{1}{8}\frac{2}{3}(4r^2+1)^{3/2}\right]_0^4$$

$$= \frac{1}{12}\left((65)^{3/2} - 1\right)$$

Q9

$$\Phi(r,\theta) = \langle r\cos\theta, r\sin\theta, r^2 \rangle \qquad 0 \le \theta \le 2\pi \quad 0 \le r \le 1$$

$$\frac{\partial \Phi}{\partial r} = \langle \cos\theta, \sin\theta, 2r \rangle$$

$$\frac{\partial \Phi}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0 \rangle$$

$$\underline{n} = \frac{\partial \Phi}{\partial r} \times \frac{\partial \Phi}{\partial \theta} = \langle -2r^2\cos\theta, -2r^2\sin\theta, r \rangle$$

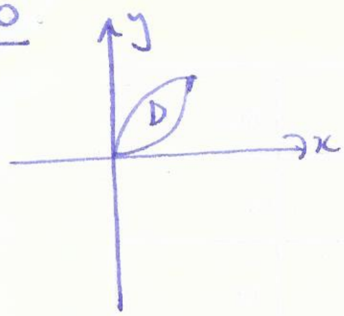
$$\int_0^{2\pi}\int_0^1 \underline{F}\cdot\underline{n}\, dr\, d\theta \qquad \underline{F} = \langle \cos\theta, \sin\theta, \cos\theta + \sin\theta \rangle$$

$$= \int_0^{2\pi}\int_0^1 \underbrace{-2r^2\cos^2\theta - 2r^2\sin^2\theta}_{-2r^2} + r\cos\theta + r\sin\theta \quad dr\, d\theta$$

$$\int_0^{2\pi} \left[-\frac{2}{3}r^3 + \frac{1}{2}r^2(\cos\theta + \sin\theta)\right]_0^1 d\theta = \int_0^{2\pi} -\frac{2}{3} + \frac{1}{2}(\cos\theta + \sin\theta) \quad d\theta.$$

$$= -\frac{4\pi}{3}$$

Q10



$$\int_C \underline{F}\cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$\underline{F} = \langle x+y,\ x^2-y \rangle$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = 2x-1$$

$$\int_0^1 \int_{\cancel{\sqrt{x}}\,x^2}^{\sqrt{x}} 2x-1 \; dy\, dx = \Big[(2x-1)y\Big]_{x^2}^{\sqrt{x}}$$

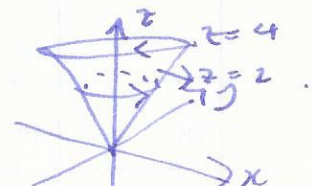
$$= (2x-1)\sqrt{x} - (2x-1)x^2$$

$$\int_0^1 2x^{3/2} - x^{1/2} - 2x^3 + x^2\, dx = \left[\frac{4}{5}x^{5/2} - \frac{2x^{3/2}}{3} - \frac{1}{2}x^4 + \frac{1}{3}x^3\right]_0^1$$

$$= \frac{4}{5} - \frac{2}{3} - \frac{1}{2} + \frac{1}{3} = \frac{4}{5} - \frac{5}{6} = \frac{24-25}{30} = -\frac{1}{30}$$

Q11

$$\iint_S \nabla\times\underline{F}\cdot d\underline{S} = \int_{\partial S} \underline{F}\cdot d\underline{s} \quad (*)$$



$z^2 = x^2 + y^2$

∂S: $z=4$: c_1 $(4\cos\theta, -4\sin\theta, 4)$ $\qquad c_1'(\theta) = (-4\sin\theta, -4\cos\theta, 0)$

$z=2$: c_2 $(2\cos\theta, 2\sin\theta, 2)$ $\qquad c_2'(\theta) = (-2\sin\theta, 2\cos\theta, 0)$

$$\int_0^{2\pi} \langle 16,\ 4\cos\theta + 16,\ 0\rangle \cdot \langle -4\sin\theta, -4\cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} -64\sin\theta - 16\cos^2\theta - 64\cos\theta\, d\theta = -16\int_0^{2\pi}\cos^2\theta\, d\theta = -8\pi$$

$$\int_0^{2\pi} \langle 4,\ 2\cos\theta + 4,\ 0\rangle \cdot \langle -2\sin\theta, 2\cos\theta, 0\rangle\, d\theta$$

$$= \int_0^{2\pi} -8\sin\theta + 4\cos^2\theta + 8\cos\theta\, d\theta = 4\int_0^{2\pi}\cos^2\theta\, d\theta = 2\pi$$

$(*) = 6\pi$