Sample midterm 2 Solutions

Q1 ① Linear approx $L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$

$f(x,y) = x^2 - 4y^2 \quad \frac{\partial f}{\partial x} = 2x \quad \frac{\partial f}{\partial y} = -8y$

at $(2,1)$: $\quad z = L(x,y) = 0 + 4(x-2) + -8(y-1) = 4x - 8y$

② Level sets of $g(x,y,z) = x^2 - 4y^2 - z \quad \nabla g = \langle 2x, -8y, -1 \rangle$

at $(2,1,0)$: $\nabla g(2,1,0) = \langle 4, -8, 0 \rangle$

so $4x - 8y \overset{-z}{\cancel{+4}} = d \qquad 8 - 8 + \overset{0}{\cancel{4}} = d \Rightarrow d = 0 \qquad 4x - 8y - z = 0.$

$z = 4x - 8y.$

Q2 $f(x,y,z) = e^{3xz} + \ln(2y+z)$

$f_x = 3z e^{3xz} + 0.$

$f_y = \frac{1}{2y+z} \cdot 2$

$f_z = 3x e^{3xz} + \frac{1}{2y+z}$

$L(x,y,z) = f(a,b,c) + f_x(a,b,c)(x-a) + f_y(a,b,c)(y-b) + f_z(a,b,c)(z-c)$

at $(1,2,4)$:

$L(x,y,z) = e^{12} + \ln(8) + 12e^{12}(x-1) + \frac{1}{4}(y-2) + \left(3e^{12} + \frac{1}{8}\right)(z-4)$

Q3 $g(x,y,z) = 4x^2 - y^2 - z \quad \nabla g = \langle 8x, -2y, -1 \rangle$

$\nabla g(1,2,0) = \langle 8, -4, -1 \rangle$ fastest way up in direction $\langle 8, -4 \rangle$.

Q4 $\frac{d}{dt}(T(\underline{r}(t))) = \nabla T(\underline{r}(t)) \cdot \underline{r}'(t)$

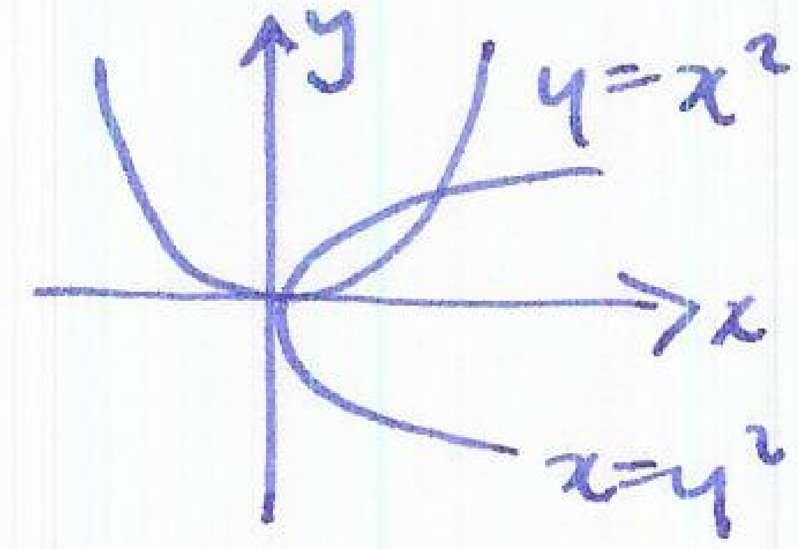
$\nabla T = 10^5 \langle -(x^2+y^2+z^2)^{-2} \cdot 2x, -(x^2+y^2+z^2)^{-2} \cdot 2y, -(x^2+y^2+z^2)^{-2} \cdot 2z \rangle.$

$\underline{r}'(t) = \langle 2, 2t, 1 \rangle$

at $t=2$ $\underline{r}(2) = \langle 4, -4, 2 \rangle$

$$\frac{d}{dt}\left(T(\underline{r}(t))\right)\Big|_{t=2} = 10^5 \frac{\langle -8, +8, -4 \rangle}{(4^2+4^2+2^2)^2} \cdot \langle 2, 4, 1 \rangle = \frac{10^5}{36^2} \cdot 12$$

Q5 a) $f(x,y) = x^3 - 3xy + y^3$



$$\left.\begin{aligned} f_x &= 3x^2 - 3y = 0 \\ f_y &= -3x + 3y^2 = 0 \end{aligned}\right\} \quad \left.\begin{aligned} y &= x^2 \\ x &= y^2 \end{aligned}\right\}$$

$x = x^4$ $\quad x(x^3-1) = 0$

$x(x-1)\underbrace{(x^2+x+1)}_{\text{no real sol}^{\underline{n}}\text{s.}} = 0$

critical points $(0,0)$, $(1,1)$

$f_{xx} = 6x$ $\quad D = f_{xx} f_{yy} - (f_{xy})^2$

$f_{xy} = -3$

$f_{yy} = 6y$ $\quad D(0,0) = 0.0 - (-3)^2 = -9 \Rightarrow$ saddle.

$D(1,1) = 6.6 - (-3)^2 = \underset{f_{xx}>0}{27} \Rightarrow$ local min

b) $f(x,y) = e^x - 2xe^y$

$f_x = e^x - 2e^y = 0$ $\quad 2e^y = 1 \Rightarrow y = \ln(1/2)$

$f_y = -2xe^y = 0 \Rightarrow x = 0$ $\quad$ critical point $(0, \ln(1/2))$

$f_{xx} = e^x$ $\quad D(0, \ln(1/2)) = 1.0 - (-2e^{\ln(1/2)})^2 < 0 \Rightarrow$ saddle.

$f_{xy} = -2e^y$

$f_{yy} = -2xe^y$

c) $f(x,y) = x\ln(x+y)$

$f_x = \ln(x+y) + \dfrac{x}{x+y} = 0$ $\quad \Rightarrow y = 1$

$f_y = \dfrac{x}{x+y} = 0 \Rightarrow x = 0$ $\quad$ critical point $(0,1)$

$$f_{xx} = \frac{1}{x+y} + \frac{(x+y)-x}{(x+y)^2}$$

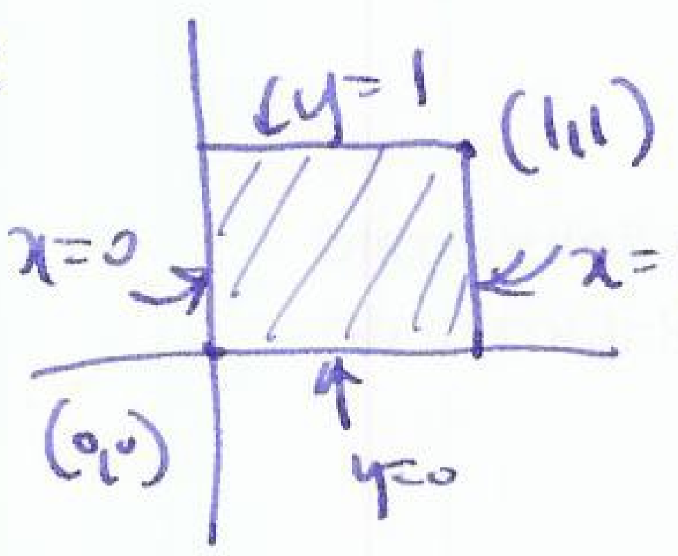
$$D(0,1) = (1+1).(0) - (1)^2 < 0$$

$\Rightarrow$ saddle.

$$f_{xy} = \frac{(x+y)-x}{(x+y)^2} = \frac{y}{(x+y)^2}$$

$$f_{yy} = \frac{-x}{(x+y)^2}$$

Q6



$$f(x,y) = 4x^2 - 2y^2$$

$$\left.\begin{aligned} f_x &= 8x = 0 \\ f_y &= -8y = 0 \end{aligned}\right\} \text{ critical point at } (0,0).$$

check boundary:

$x=0$: $f(0,y) = -2y^2$ $\quad \frac{df(0,y)}{dy} = -8y \quad y=0.$ critical point on edge.

$x=1$: $f(1,y) = 4-2y^2$ $\quad \frac{d}{dy}(f(1,y)) = -8y \quad y=0.$

$y=0$: $f(x,0) = 4x^2$ $\quad \frac{d}{dx}(f(x,0)) = 8x \quad x=0$

$y=1$: $f(x,1) = 4x^2-2$ $\quad \frac{d}{dx}(f(x,1)) = 8x \quad x=0$

check corners:

$f(0,0)$	$f(0,1)$	$f(1,0)$	$f(1,1)$
0	-2	4	2
min		max	

Q7 max $f(x,y) = x^2y + x + y$ subject to $g(x,y) = xy = 4$.

$\nabla f = \langle 2xy+1, x^2+1 \rangle \qquad \nabla g = \langle y, x \rangle$

$\nabla f = \lambda \nabla g$:

$$\left.\begin{aligned} 2xy+1 &= \lambda y \\ x^2+1 &= \lambda x \end{aligned}\right\} \quad \frac{2xy+1}{x^2+1} = \frac{y}{x} \quad \frac{\frac{2x\cdot 4}{x}+1}{x^2+1} = \frac{4}{x^2}.$$

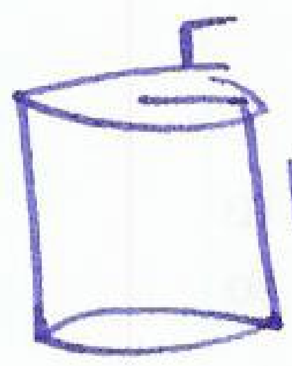
$xy = 4. \qquad y = 4/x$

$$\frac{9}{x^2+1} = \frac{4}{x^2} \qquad 9x^2 = 4x^2+1 \qquad 5x^2 = 1 \qquad x = \pm 1/\sqrt{5},\ y = \pm 4/\sqrt{5}$$

$$f\left(\tfrac{1}{\sqrt5},\tfrac{4}{\sqrt5}\right) = \frac{1}{5}\cdot\frac{4}{\sqrt5}+\frac{1}{\sqrt5}+\frac{4}{\sqrt5} \quad \text{max}$$

$$f\left(-\tfrac{1}{\sqrt5},-\tfrac{4}{\sqrt5}\right) = \frac{1}{\sqrt5}\cdot\frac{4}{\sqrt5}-\frac{1}{\sqrt5}-\frac{4}{\sqrt5} \quad \text{min}$$

Q8



$V = \pi r^2 h$

$A = 2\pi r h + 2\pi r^2$

min $A(r,h) = 2\pi r h + 2\pi r^2$ subject to $V(r,h) = \pi r^2 h = V$

$\nabla A = \langle 2\pi h + 4\pi r, 2\pi r \rangle \qquad \nabla V = \langle 4\pi r h, \pi r^2 \rangle$

$\nabla A = \lambda \nabla V$: $\quad 2\cancel{\pi} h + 4\cancel{\pi} r = \lambda 4\cancel{\pi} r h \qquad 2h + 4r = \frac{2}{r} 4rh$

$2\cancel{\pi}\cancel{r} = \lambda\cancel{\pi} r^{\cancel{2}} \qquad 2 = \lambda r \qquad 4r = 6h$

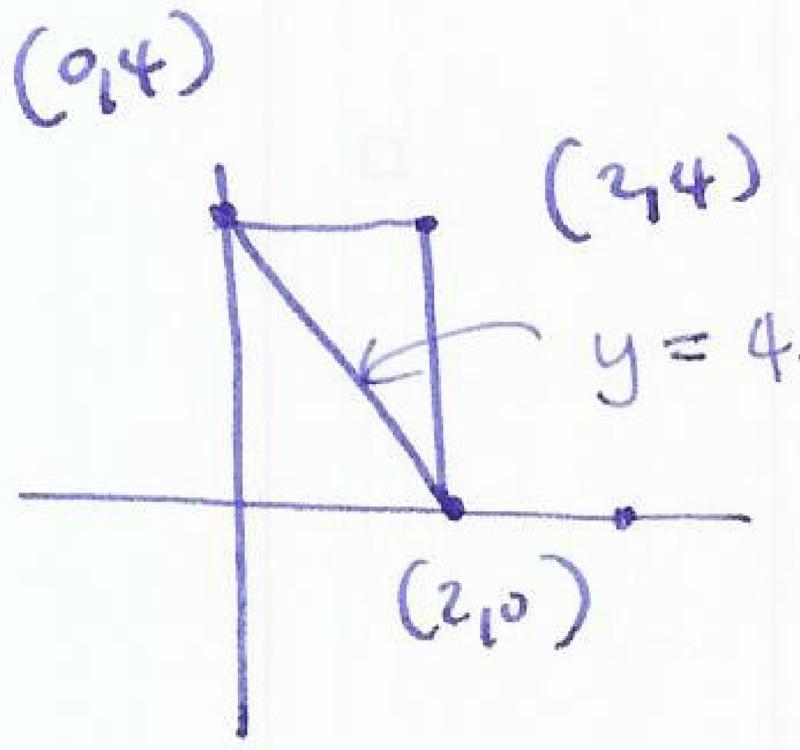
$\pi r^2 h = V \qquad 2r = 3h$

$\boxed{\text{choose } h = \frac{2}{3} r}$

where $\pi r^2 \frac{2}{3} r = V \qquad r^3 = \frac{3V}{2\pi}$

$$\boxed{r = \sqrt[3]{\frac{3V}{2\pi}} \qquad h = \frac{2}{3}\sqrt[3]{\frac{3V}{2\pi}}}$$

Q9

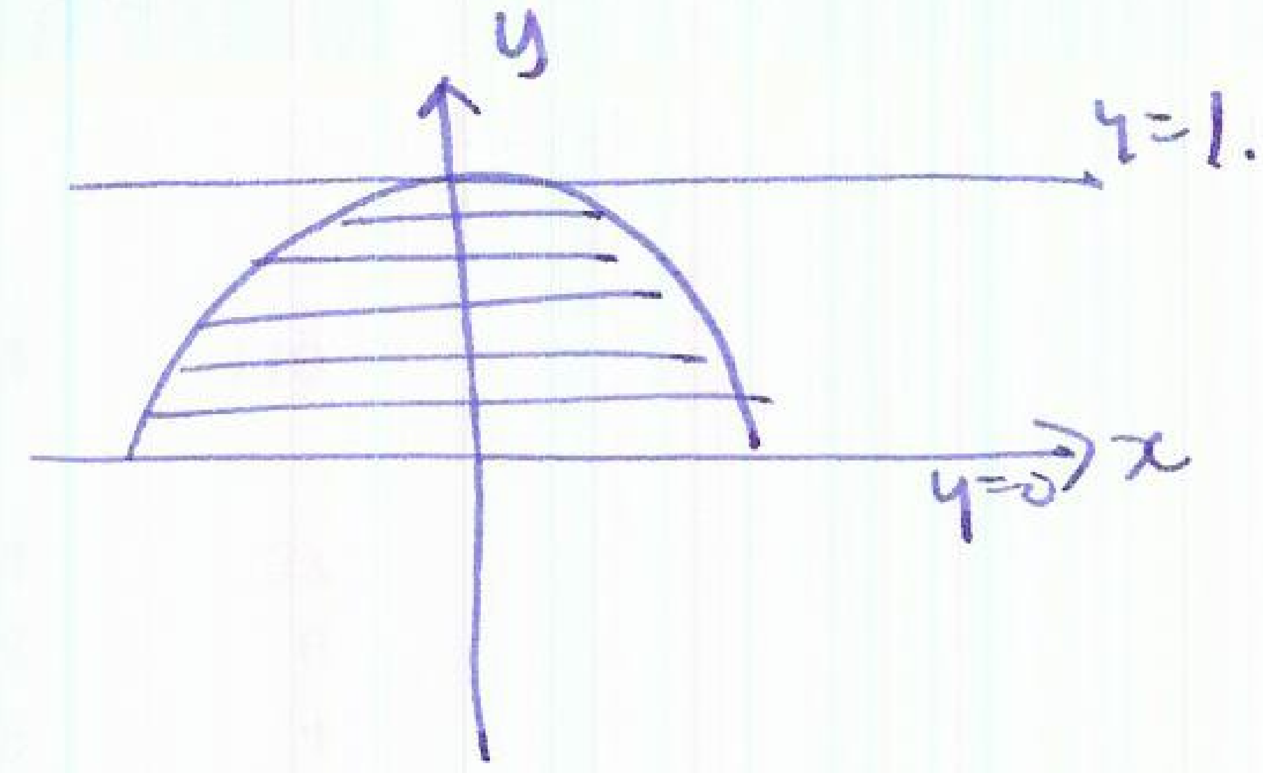


$$\int_0^2 \underbrace{\int_{4-2x}^{4} xy \, dy}\, dx$$

$$\left[\frac{1}{2} x y^2\right]_{4-2x}^{4} = 8x - \frac{1}{2}x(4-2x)^2$$

$$\int_0^2 8x - 8x + 8x^2 - 2x^3 \, dx = \left[\frac{8}{3}x^3 - \frac{1}{2}x^4\right]_0^2 = \frac{64}{3} - 8.$$

Q10 $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \frac{y}{(1+x^2+y^2)^2}\,dx\,dy$



$x=\sqrt{1-y^2} \Leftrightarrow x^2=1-y^2 \Leftrightarrow x^2+y^2=1$

$$\int_{-1}^{1} \underbrace{\int_0^{\sqrt{1-x^2}} \frac{y}{(1+x^2+y^2)^2}\,dy}\,dx$$

$$\left[-\frac{1}{2}(1+x^2+y^2)^{-1}\right]_0^{\sqrt{1-x^2}} = -\frac{1}{2}(1+x^2+1-x^2)^{-1}+\frac{1}{2}(1+x^2)^{-1}$$

$$\int_{-1}^{1} -\frac{1}{4}+\frac{1}{2(1+x^2)}\,dx = \left[-\frac{1}{4}x+\frac{1}{2}\tan^{-1}(x)\right]_{-1}^{1} = -\frac{1}{4}+\frac{1}{2}\tan^{-1}(1) - \left(\frac{1}{4}+\frac{1}{2}\tan^{-1}(-1)\right)$$

$$= -\frac{1}{2}+\tan^{-1}(1) = \frac{\pi}{4}-\frac{1}{2}.$$