Math 233 Calculus 3 Spring 13 Final b

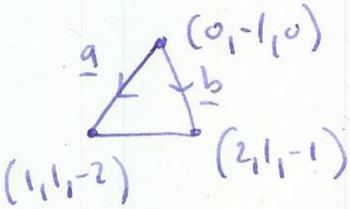
Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) (10 points) Find the equation of of the pane containing the points $(0, -1, 0)$, $(1, 1, -2)$ and $(2, 1, -1)$.



$\underline{a} = (1, 2, -2)$

$\underline{b} = (2, 2, -1)$

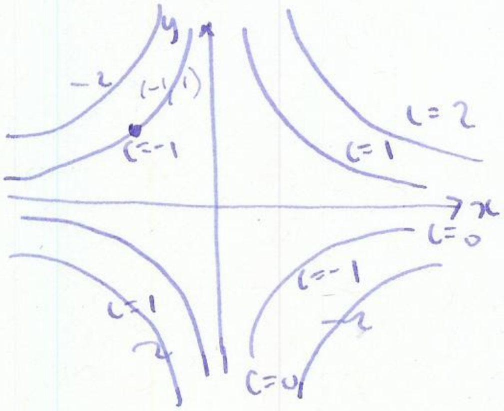
$\underline{n} = \underline{a} \times \underline{b} = (2, -3, -2)$

$2x - 3y - 2z = d$

at $(0, -1, 0)$: $3 = d$.

equation of plane: $2x - 3y - 2z = 3$

(2) Sketch the level sets of the function $f(x, y) = xy$, and calculate the gradient vector at the point $(-1, 1)$. Use this to find the tangent line to $xy = -1$ at the point $(-1, 1)$.



$$\nabla f = \langle y, x \rangle$$

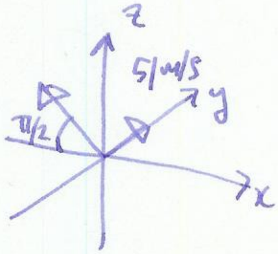
$$\nabla f(-1,1) = \langle 1, -1 \rangle$$

$$x - y = c \qquad \text{wher } c = -1-1 = -2$$

$$x - y = -2$$

$$y = x + 2$$

(3) You are driving due north at 5m/s, and you throw a tennis ball due west at 10m/s at an angle of $\cancel{\pi/2}$ with the horizontal. How fast is the tennis ball going when it hits the ground?



$\pi/4.$

$$\underline{x}''(t) = \langle 0, 0, -10 \rangle$$

$$\underline{x}'(t) = \langle 0, 0, -10 \rangle t + \underline{v}_0$$

$$\underline{x}(t) = \langle 0, 0, -10 \rangle \tfrac{1}{2}t^2 + \underline{v}_0 t + \underline{x}_0$$

$\underline{x}_0 = \langle 0, 0, 0 \rangle$

$\underline{v}_0 = \langle 0, 5, 0 \rangle + \langle -10\cos(\tfrac{\pi}{2}), 0, 10\sin(\tfrac{\pi}{2}) \rangle = \langle -5\sqrt{2}, 5, 5\sqrt{2} \rangle$

$\underline{x}(t) = \langle -5\sqrt{2}t, 5t, 5\sqrt{2}t - 5t^2 \rangle.$

hits ground at $z = 0$: $\quad 5\sqrt{2}t - 5t^2 = 0 \quad 5t(\sqrt{2} - \cancel{5}t) = 0$

$t = \sqrt{2}$

velocity at $t = \sqrt{2}$: $\quad \langle -5\sqrt{2}, 5, 5\sqrt{2} - 10\sqrt{2} \rangle = \langle -5\sqrt{2}, 5, -5\sqrt{2} \rangle$

speed $= \|\underline{x}'(\sqrt{2})\| = 5\sqrt{2+1+2} = 5\sqrt{5}.$

(4) Find the critical points of $f(x, y) = xe^{-2x} + y^2e^{-2x}$ and use the second derivative test to classify them.

$f_x = e^{-2x} + x \cdot -2e^{-2x} + -2y^2e^{-2x} = 0$

$f_y = 2ye^{-2x} = 0 \Rightarrow y=0$

$e^{-2x}(1-2x) = 0 \Rightarrow x = \frac{1}{2}.$

critical point $(\frac{1}{2}, 0)$.

$f_{xx} = -2e^{-2x} + -2e^{-2x} + x\,4e^{-2x} + 4y^2e^{-2x}$

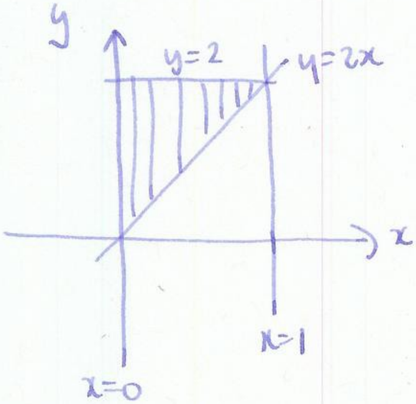
$f_{xy} = -4ye^{-2x}$

$f_{yy} = 2e^{-2x}$

$D = f_{xx}f_{yy} - (f_{xy})^2$

$D(\frac{1}{2}, 0) = (-2e^{-1})(2e^{-1}) - (0) = -4e^{-1} < 0 \Rightarrow$ saddle

(5) Change the order of integration to evaluate $\int_0^1 \int_{2x}^2 e^{-y^2}\, dy\, dx$.



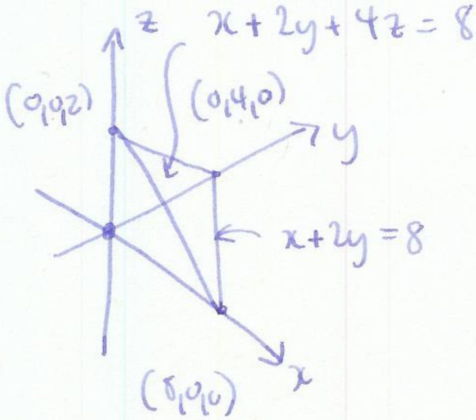
$$\int_0^2 \underbrace{\int_0^{\frac{1}{2}y} e^{-y^2}\, dx}\, dy$$

$$\left[xe^{-y^2}\right]_0^{\frac{1}{2}y} = \frac{1}{2}ye^{-y^2}.$$

$$\int_0^2 \frac{1}{2}ye^{-y^2}\, dy = \left[-\frac{1}{4}e^{-y^2}\right]_0^2$$

$$= +\frac{1}{4}\left(1-e^{-4}\right).$$

(6) Use a triple integral to find the volume of the tetrahedron formed with vertices $(0,0,0)$, $(8,0,0)$, $(0,4,0)$ and $(0,0,2)$.



$$\int_0^8 \int_0^{4-\frac{x}{2}} \int_0^{2-\frac{y}{2}-\frac{x}{4}} 1 \, dz\, dy\, dx.$$

$$\left[z \right]_0^{2-\frac{y}{2}-\frac{x}{4}} = 2 - \tfrac{1}{2}y - \tfrac{1}{4}x.$$

$$\left[2y - \tfrac{1}{4}y^2 - \tfrac{1}{4}xy \right]_0^{4-\frac{x}{2}}$$

$$8 - x - \tfrac{1}{4}\left(4-\tfrac{x}{2}\right)^2 - \tfrac{1}{4}x\left(4-\tfrac{x}{2}\right)$$

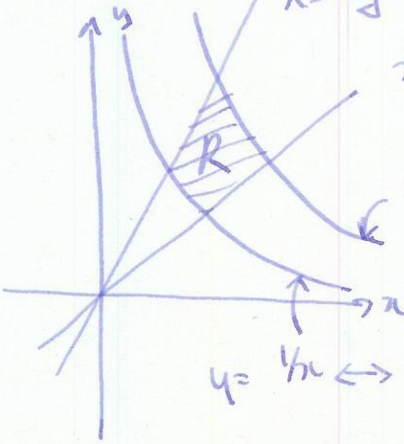
$$8 - x - \tfrac{1}{4}\left(16 - 4x + \tfrac{x^2}{4}\right) - x + \tfrac{x^2}{8}$$

$$4 + \cancel{\tfrac{}{8}}\ -x + \frac{x^2}{16}$$

$$\left[4x + \cancel{\tfrac{}{16}}\ -\tfrac{1}{2}x^2 + \frac{x^3}{2\cdot 16} \right]_0^8$$

$$= 4\cdot 8 - \cancel{4}\ 8\cdot 4 + \frac{4\cdot 8}{3} = \cancel{}\ \frac{32}{3}.$$

(7) Use the change of variable $x = uv$, $y = u/v$ to evaluate $\int\int_R \frac{1}{y}\, dx\, dy$, where R is the region bounded by the curves $y = 1/x$, $y = 2/x$, $x = y$ and $x = 2y$.

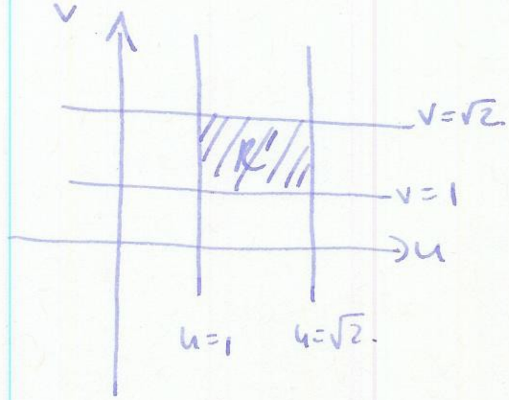


$x = 2y \iff uv = \frac{2u}{v} \iff v^2 = 2$

$x = y \iff uv = \frac{u}{v} \iff v^2 = 1.$

$y = \frac{2}{x} \iff \frac{u}{v} = \frac{2}{uv} \iff u^2 = 2.$

$y = 1/x \iff \frac{u}{v} = \frac{1}{uv} \iff u^2 = 1$



$$\int\int_{R'} \frac{v}{u} J \, du\, dv$$

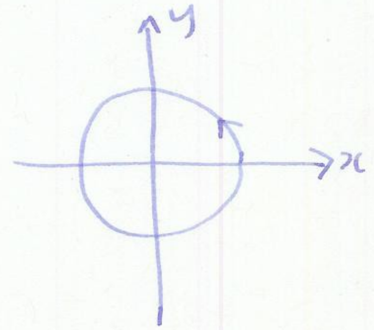
$$J = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} \right| = \left| -\frac{u}{v} - \frac{u}{v} \right| = \frac{2u}{v}.$$

$$\int_1^{\sqrt{2}} \underbrace{\int_1^{\sqrt{2}} \frac{v}{u} \cdot \frac{2u}{v}\, du}\, dv = \quad \left[2u\right]_1^{\sqrt{2}} = 2(\sqrt{2}-1)$$

$$\left[2(\sqrt{2}-1)v\right]_1^{\sqrt{2}} = 2(\sqrt{2}-1)^2$$

(8) Let C be the unit circle with anticlockwise orientation, and let $\mathbf{F} = \langle x^2 y, \sin(1/y)\rangle$. Use Green's Theorem to evaluate

$$\int_C \mathbf{F}\, ds.$$



$$\int_C \underline{F}\cdot ds = \iint_R \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA = \iint_R 0 - x^2\, dA$$

polars:
$x = r\cos\theta$
$y = r\sin\theta$

$$\int_0^{2\pi}\int_0^1 -r^2\cos^2\theta\; r\, dr\, d\theta$$

$$\int_0^1 -r^3\, dr = \left[-\frac{1}{4}r^4\right]_0^1 = -\frac{1}{4}.$$

$$\int_0^{2\pi} \cos^2\theta\, d\theta = \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta\, d\theta = \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi} = \pi$$

answer: $-\frac{\pi}{4}$

$-xy$

(9) Let $\mathbf{F} = \langle 2xy, e^{-y^2}, xy\rangle$. Let S be the part of the cylinder $x^2 + y^2 = 1$, with $0 \le z \le 1$, with the outward pointing normal. Use Stokes' Theorem to evaluate $\int_{\partial S} \mathbf{F}\, ds$.

$$\int_{\partial C} \underline{F}\, d\underline{s} = \iint_S \operatorname{curl}(\underline{F})\, dS$$

$$\operatorname{curl}(\underline{F}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & e^{-y^2} & -xy \end{vmatrix} = \langle -x, +y, -2x \rangle$$

$$T(\theta, z) = (\cos\theta, \sin\theta, z) \qquad 0 \le \theta \le 2\pi \qquad 0 \le z \le 1$$

$$\frac{\partial T}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$$

$$\frac{\partial T}{\partial z} = (0, 0, 1)$$

$$\underline{n} = (\cos\theta, \sin\theta, 0)$$

$$\int_0^1 \int_0^{2\pi} \underbrace{(-\cos\theta, \sin\theta, -2\cos\theta)\cdot(\cos\theta, \sin\theta, 0)}_{= -\cos^2\theta + \sin^2\theta \,=\, -\cos 2\theta}\, d\theta\, dz$$

$$\int_0^{2\pi} \underset{-\cos 2\theta}{v}\, d\theta = \cancel{2\pi}\ 0 \qquad \int_0^1 dz = 1 \qquad \text{answer} : \cancel{2\pi}\ 0.$$

(10) Let W be the hemisphere $x^2 + y^2 + z^2 \leq 1$ with $0 \leq z \leq 1$, and let $\mathbf{F} = \langle x, y, z + e^{-x^2} \rangle$. Use the divergence theorem to evaluate $\displaystyle\iint_{\partial W} \mathbf{F}\, d\mathbf{S}$.

$$\iint_{\partial W} \underline{F}\, dS = \iiint_W \overset{\text{div}}{\cancel{\text{curl}}}(F)\, dV$$

$$\text{div}(\underline{F}) = 1 + 1 + 1 = 3$$

$$\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} 3\, \rho^2 \sin\phi \, d\phi\, d\theta\, d\rho. \quad (*)$$

$$\int_0^{\pi/2} \sin\phi \, d\phi = \Big[-\cos\phi\Big]_0^{\pi/2} = -0 + 1 = 1$$

$$\int_0^1 \rho^2 \, d\rho = \Big[\tfrac{1}{3}\rho^3\Big]_0^1 = \frac{1}{3}.$$

$$\int_0^{2\pi} 1 \, d\theta = 2\pi$$

$$(*) = 3 \cdot 1 \cdot \frac{1}{3} \cdot 2\pi = 2\pi.$$