Math 233 Calculus 3 Spring 13 Final a

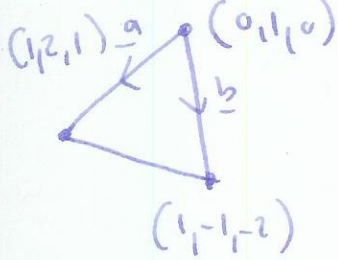
Name: Solutions

- Do any 8 of the following 10 questions.
- You may use a calculator without symbolic algebra capabilities, but no notes.

1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
	80	

Final	
Overall	

(1) (10 points) Find the equation of of the pane containing the points $(0, 1, 0)$, $(1, 2, 1)$ and $(1, -1, -2)$.



$\underline{a} = (1, 1, 1)$

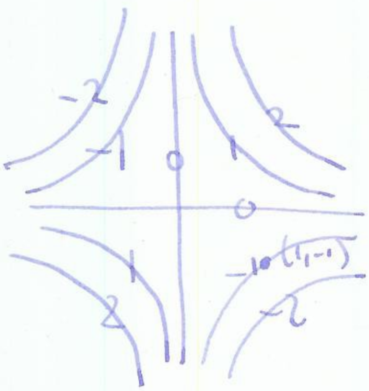
$\underline{b} = (1, -2, -2)$

$\underline{n} = \underline{a} \times \underline{b} = (0, 3, -3) = 3(0, 1, -1)$

$0x + \cancel{3}y - \cancel{3}z = d$ at $(0,1,0) \Rightarrow d = \cancel{3}\,1$

equation : $y - z = \cancel{3}\,1$

(2) Sketch the level sets of the function $f(x, y) = xy$, and calculate the gradient vector at the point $(1, -1)$. Use this to find the tangent line to $xy = -1$ at the point $(1, -1)$.



$$\nabla f = \langle y, x \rangle$$

$$\nabla f(1,-1) = \langle -1, 1 \rangle.$$

$$-x + y = d \quad \text{at } (1,-1)$$

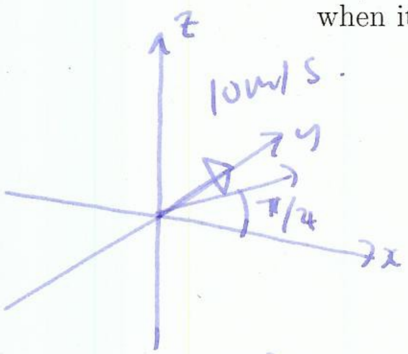
$$-1 - 1 = d \Rightarrow d = -2.$$

$$-x + y = -2$$

$$y = x - 2.$$

$\pi/4$

(3) You are driving due north at 10m/s, and you throw a tennis ball due east at 5m/s at an angle of $\cancel{\pi/2}$ with the horizontal. How fast is the tennis ball going when it hits the ground?



$$\underline{x}''(t) = \langle 0, 0, -10 \rangle$$

$$\underline{x}'(t) = \langle 0, 0, -10 \rangle t + \underline{v}_0$$

$$\underline{x}(t) = \langle 0, 0, -10 \rangle \tfrac{1}{2} t^2 + \underline{v}_0 t + \underline{x}_0$$

$x_0 = (0, 0, 0)$

$\underline{v}_0 = \langle 0, 10, 0 \rangle + \langle 5 \cos \frac{\pi}{4}, 0, 5 \sin \frac{\pi}{4} \rangle. = \langle \frac{5\sqrt{2}}{2}, 10, \frac{5\sqrt{2}}{2} \rangle$

$\underline{x}(t) = \langle \frac{5\sqrt{2}}{2} t, 10t \quad \frac{5\sqrt{2}}{2} t - 5t^2 \rangle$

hits ground at $z = 0$: $\quad 5t\left(\frac{\sqrt{2}}{2} - t\right) = 0 \quad t = 0, \; t = \frac{\sqrt{2}}{2}.$

velocity at $t = \frac{\sqrt{2}}{2}$: $\quad \underline{x}'\left(\frac{\sqrt{2}}{2}\right) = \langle \frac{5\sqrt{2}}{2}, 10, \frac{5\sqrt{2}}{2} - 10\frac{\sqrt{2}}{2} \rangle.$

$$= \langle \frac{5\sqrt{2}}{2}, 10, -\frac{5\sqrt{2}}{2} \rangle.$$

speed $\|x'\left(\frac{\sqrt{5}}{2}\right)\| = 5\sqrt{\frac{1}{2} + 4 + \frac{1}{2}} = 5\sqrt{5}$

(4) Find the critical points of $f(x,y) = x^2e^{-2y} + ye^{-2y}$ and use the second derivative test to classify them.

$f_x = 2xe^{-2y} = 0 \Rightarrow x = 0$

$f_y = -2x^2e^{-2y} + e^{-2y} + -2ye^{-2y} = 0$

$e^{-2y}(1-2y) = 0 \Rightarrow y = \frac{1}{2}$

critical point $(0, \frac{1}{2})$.

$f_{xx} = 2e^{-2y}$

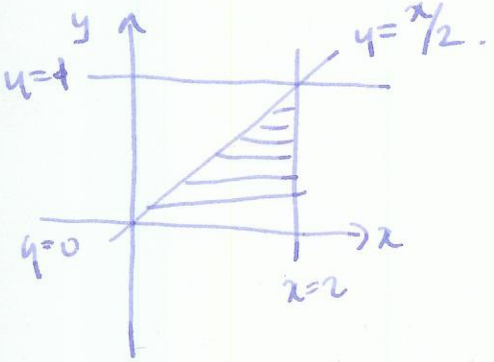
$f_{xy} = -4xe^{-2y}$

$f_{yy} = 4x^2e^{-2y} - 2e^{-2y} - 2e^{-2y} + 4ye^{-2y}$

$D = f_{xx} f_{yy} - (f_{xy})^2$

$D(0, \frac{1}{2}) = (2e^{-1})(-2e^{-1}) - (0) = -4e^{-2} < 0 \Rightarrow$ saddle.

(5) Change the order of integration to evaluate $\int_0^1 \int_{2y}^2 e^{-x^2}\, dx\, dy$.

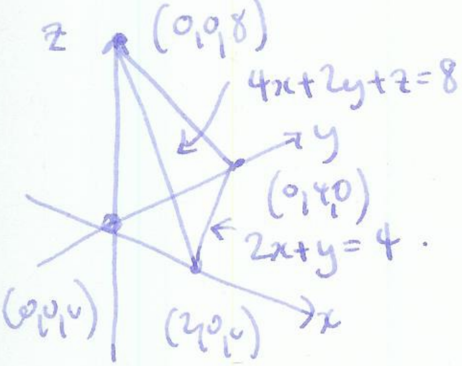


$$\underbrace{\int_0^2 \int_0^{x/2} e^{-x^2}\, dy}\, dx$$

$$\underbrace{\left[y e^{-x^2} \right]_0^{x/2} = \frac{x}{2} e^{-x^2}.}$$

$$\left[-\frac{1}{4} e^{-x^2} \right]_0^2 = \frac{1}{4}\left(1 - e^{-4}\right)$$

(6) Use a triple integral to find the volume of the tetrahedron formed with vertices $(0,0,0)$, $(2,0,0)$, $(0,4,0)$ and $(0,0,8)$.



$$\int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y} 1 \, dz \, dy \, dx$$

$$\Big[z \Big]_0^{8-4x-2y} = 8-4x-2y$$

$$\Big[8y - 4xy - y^2 \Big]_0^{4-2x}$$

$$8(4-2x) - 4x(4-2x) - (4-2x)^2.$$

$$32 - 16x - 16x + 8x^2 - 16 + 16x - 4x^2$$

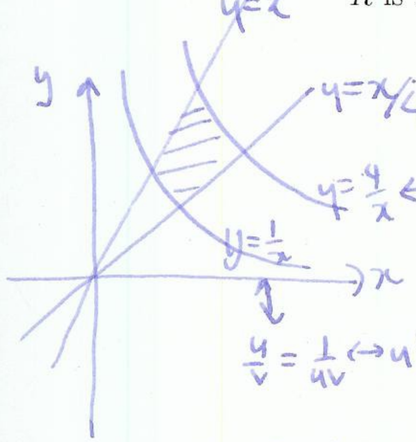
$$16 - 16x + 4x^2.$$

$$\Big[16x - 8x^2 + \frac{4}{3}x^3 \Big]_0^2$$

$$= 32 - 32 + \frac{4.8}{3} = \frac{32}{3}$$

(7) Use the change of variable $x = uv$, $y = u/v$ to evaluate $\displaystyle\int\int_R \frac{1}{y}\,dx\,dy$, where R is the region bounded by the curves $y = 1/x$, $y = 4/x$, $x = y$ and $x = 4y$.

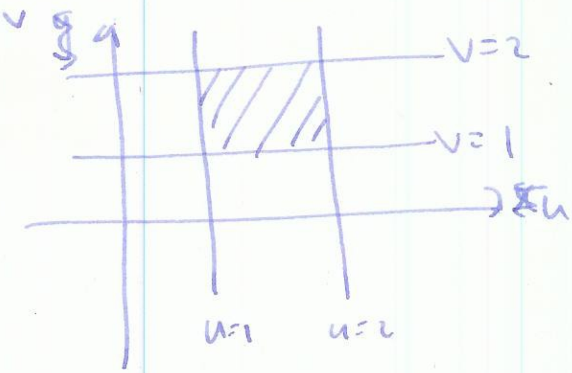
$uv = \frac{u}{v} \leftrightarrow v^2 = 1$



$y = x/4 \leftrightarrow \frac{uv}{4} = \frac{u}{v} \leftrightarrow v^2 = 4$

$y = \frac{4}{x} \leftrightarrow \frac{u}{v} = \frac{4}{uv} \leftrightarrow u^2 = 4$

$\frac{u}{v} = \frac{1}{uv} \leftrightarrow u^2 = 1$



$$J = \left| \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} \right| = \left| \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} \right| = \left| -\frac{u}{v} - \frac{u}{v} \right| = \frac{2u}{v}$$

$$\underbrace{\int_1^2 \underbrace{\int_1^2 \frac{v}{u} \cdot \frac{2u}{v}\, du}_{[2u]_1^2 = 2}\, dv}_{[2v]_1^2 = 2}$$

$$\left[2u\right]_1^2 = 2$$

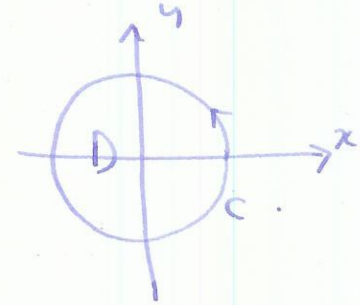
$$\left[2v\right]_1^2 = 2.$$

(8) Let C be the unit circle with anticlockwise orientation, and let $\mathbf{F} = \langle x^2y, e^{-y^2}\rangle$. Use Green's Theorem to evaluate

$$\int_C \mathbf{F}\, ds. = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$y \quad y$$

$$0 - x^2$$



polars: $\displaystyle\int_0^{2\pi}\int_0^1 -r^2\cos^2\theta\; r\,dr\,d\theta$

$$\int_0^1 -r^3\,dr = \left[-\frac{1}{4}r^4\right]_0^1 = -\frac{1}{4}.$$

$$\int_0^{2\pi}\cos^2\theta\,d\theta = \int_0^{2\pi}\frac{1}{2}-\frac{1}{2}\cos 2\theta\,d\theta = \left[\frac{\theta}{2}-\frac{1}{4}\cos 2\theta\right]_0^{2\pi} = \pi$$

answer : $-\frac{\pi}{4}$

$-xy$

(9) Let $\mathbf{F} = \langle 2xy, \sin(1/y), xy\rangle$. Let S be the part of the cylinder $x^2+y^2=1$, with $0 \le z \le 1$, with the outward pointing normal. Use Stokes' Theorem to evaluate $\displaystyle\int_{\partial \cancel{C} S} \mathbf{F}\, d\mathbf{s}$.

$$\int_{\partial S} \underline{F}\cdot d\underline{s} = \iint_S \operatorname{curl}(\underline{F})\, dS$$

$$\operatorname{curl}(F) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & \sin(\frac{1}{y}) & -xy \end{vmatrix} = \langle -x, +y, -2x \rangle$$

$$T(\theta, z) = (\cos\theta, \sin\theta, z) \qquad 0 \le \theta \le 2\pi \qquad 0 \le z \le 1$$

$$\frac{\partial T}{\partial \theta} = (-\sin\theta, \cos\theta, 0)$$

$$\frac{\partial T}{\partial z} = (0, 0, 1)$$

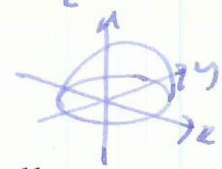
$$\underline{n} = \frac{\partial T}{\partial \theta} \times \frac{\partial T}{\partial z} = (\cos\theta, \sin\theta, 0)$$

$$\int_0^1 \int_0^{2\pi} \underbrace{(-\cos\theta, \sin\theta, -2\cos\theta)\cdot(\cos\theta, \sin\theta, 0)}_{-\cos^2\theta + \sin^2\theta = -\cos 2\theta} \, d\theta\, dz$$

$$\int_0^{2\pi} \underset{-\cos 2\theta}{\cancel{1}}\, d\theta = \left[\cancel{\theta}\right]_0^{2\pi} = \cancel{2\pi}\ 0$$

$$\int_0^1 1\, dz = [z]_0^1 = 1$$

answer $\cancel{2\pi}$ 0



(10) Let W be the hemisphere $x^2 + y^2 + z^2 \leq 1$ with $0 \leq z \leq 1$, and let $\mathbf{F} = \langle x, y + e^{-z^2}, z\rangle$. Use the divergence theorem to evaluate $\displaystyle\iint_{\partial W} \mathbf{F}\, d\mathbf{S}$. $= \displaystyle\iiint_W \operatorname{div}(\mathbf{F})\, dV$

$\operatorname{div}(\mathbf{F}) = 1 + 1 + 1 = 3$

use sphericals: $\displaystyle\int_0^1 \int_0^{2\pi} \int_0^{\pi/2} 3 \quad \rho^2 \sin\phi \; d\phi\, d\theta\, d\rho.$

$\displaystyle\int_0^{\pi/2} \sin\phi \, d\phi = \left[-\cos\phi\right]_0^{\pi/2} = 0 - (-1) = 1$

$\displaystyle\int_0^{2\pi} 1 \, d\theta = \left[\theta\right]_0^{2\pi} = 2\pi$

$\displaystyle\int_0^1 3\rho^2 d\rho = \left[\tfrac{3}{3}\rho^3\right]_0^1 = 1$

answer 2π.