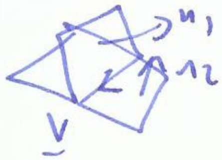
sample final solutions

Q1



$\underline{v} = \underline{n}_1 \times \underline{n}_2$

$\underline{n}_1 = \langle 2, -3, -1 \rangle$

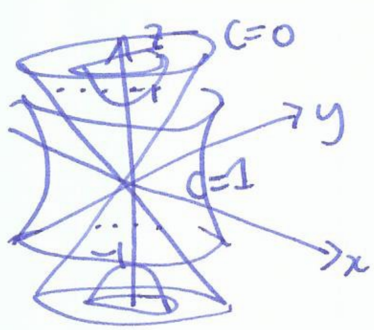
$\underline{n}_2 = \langle 2, -1, -1 \rangle$

$\underline{v} = \langle 2, 0, 4 \rangle$

find point on line : $z=0$ $\left.\begin{array}{l} 2x-3y=-2 \\ 2x-y=4 \end{array}\right\} -2y=-6 \quad y=3, \; x=7/2$

line: $(\frac{7}{2}, 3, 0) + t(2, 0, 4)$

Q2 $f(x,y,z) = x^2+y^2-z^2 = c$



$\nabla f = \langle 2x, 2y, -2z \rangle$

$\nabla f(2,2,2) = \langle 4, 4, -4 \rangle$

tangent plane to $x^2+y^2 = z^4+4$

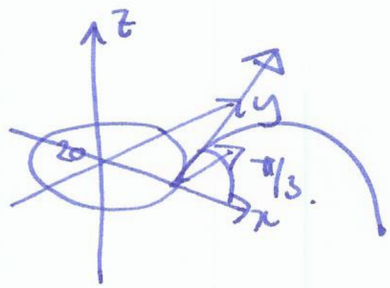
at $(2,2,2)$ is $4x+4y-4z=d$

$8+8-8=d \quad d=8$

$4x+4y-4z=8$

$x+y-z=2$

Q3



$\ddot{x}(t) = \langle 0, 0, -10 \rangle$

$\dot{x}(t) = \langle 0, 0, -10 \rangle t + \underline{v}_0$

$x(t) = \langle 0, 0, -5 \rangle t^2 + \underline{v}_0 t + \underline{x}_0$.

$\underline{x}_0 = \cancel{\langle 0, 0} \; \langle 20, 0, 0 \rangle$

$\underline{v}_0 = \langle 0, 5, 0 \rangle + \langle 10\underbrace{\cos(\tfrac{\pi}{3})}_{1/2}, 0, 10\underbrace{\sin(\tfrac{\pi}{3})}_{\sqrt{3}/2} \rangle$

$\underline{x}(t) = \langle 0, 0, -5 \rangle t^2 + \langle 5, 5, \frac{5\sqrt{3}}{2} \cancel{t} \rangle t + \langle 20, 0, 0 \rangle$.

$= \langle 20+5t, 5t, -5t^2 + \frac{5\sqrt{3}}{2} t \rangle$

$z=0$: $t(-5t + \frac{5\sqrt{3}}{2}) = 0 \quad t=0$ or $t = \frac{\cancel{\sqrt{3}}}{\cancel{10}}\sqrt{3}$.

lands at $\underline{x}(\sqrt{3}) = \langle 20+5\sqrt{3}, 5\sqrt{3}, 0 \rangle$

②

Q4 $f(x,y) = 2x^2+y^2-4y+3$

a) $\frac{\partial f}{\partial x} = 4x \quad \frac{\partial f}{\partial y} = 2y-4$

$\frac{\partial f}{\partial x} = 0 \Rightarrow x=0$

$\frac{\partial f}{\partial y} = 0 \Rightarrow y=2$

critical point $(0,2)$

$\frac{\partial^2 f}{\partial x^2} = 4 \quad \frac{\partial^2 f}{\partial y^2} = 2 \quad \frac{\partial^2 f}{\partial x \partial y} = 0$

$D(0,2) = 4.2-0 = 8>0$

$f_{xx}(0,2) > 0 \Rightarrow$ minimum.

b) $\underset{min}{\cancel{max}}\ f(x,y)$ subject to $x^2+y^2=9$

$\nabla f = \lambda \nabla g \quad \nabla f = \langle 4x, 2y-4 \rangle \quad \nabla g = \langle 2x, 2y \rangle$.

$4x = \lambda 2x \Rightarrow \lambda = 2$ or $x=0 \Rightarrow y = \pm 3.$ $\quad (0,3) \quad (0,-3)$

$2y-4 = \lambda 2y \quad \hookrightarrow y=-2 \quad x = \pm\sqrt{5} \quad (-2,\sqrt{5}) \quad (-2,+\sqrt{5}).$

$x^2+y^2=9$

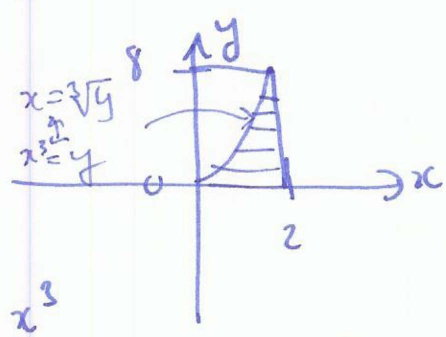
c) $f(0,2) = 4-8+3 = -1$ min.

$f(0,3) = 9-12+3 = 0$

$f(0,-3) = 9+12+3 = 24$ max.

$\left.\begin{array}{l} f(-2,\sqrt{5}) = 8+5-4\sqrt{5}+3 = 16-4\sqrt{3}. \\ f(-2,-\sqrt{5}) = 8+5+4\sqrt{5}+3 = 16+4\sqrt{3}. \end{array}\right\} < 24.$

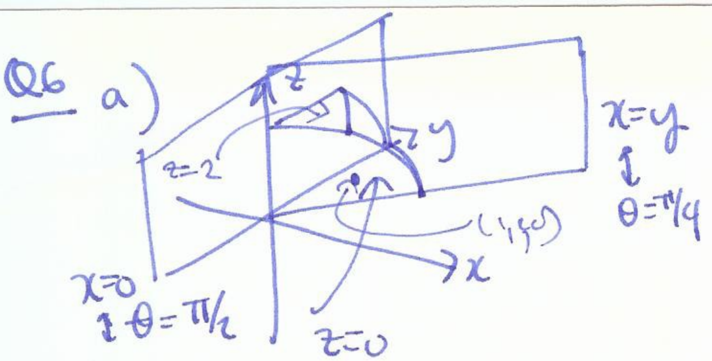
Q5 $\int_0^8 \int_{\sqrt[3]{y}}^{2} \sin(x^4)\,dx\,dy$



$\int_0^2 \int_0^{x^3} \sin(x^4)\,dy\,dx = \left[y\sin(x^4)\right]_0^{x^3} = x^3\sin(x^4)$

$\int_0^2 x^3\sin(x^4)\,dx = \left[-\cos(x^4)\frac{1}{4}\right]_0^2 = \frac{1}{4} - \frac{1}{4}\cos(8).$

Q6 a)



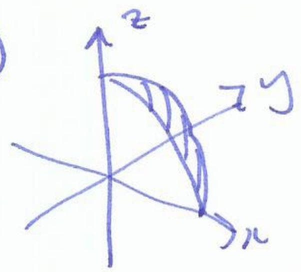
cylindricals. $x = r\cos\theta$
$y = r\sin\theta$
$z = z$.

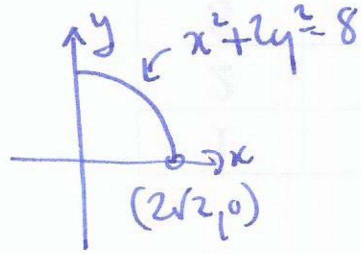
$z=2$ intersects $x^2+y^2+z^2=9$
in $x^2+y^2=5 \leftrightarrow r^2=5$.

$$\int_{\pi/4}^{\pi/2}\int_0^{\sqrt5}\int_0^2 f(-)\,dz\,dr\,d\theta + \int_{\pi/4}^{\pi/2}\int_{\sqrt5}^{3}\int_0^{\sqrt{9-r^2}} f(-)\,dz\,dr\,d\theta$$

b) $$\int_0^{2\pi}\int_0^2\int_0^{3-x/4-y/2} f(-)\,dz\,dr\,d\theta$$

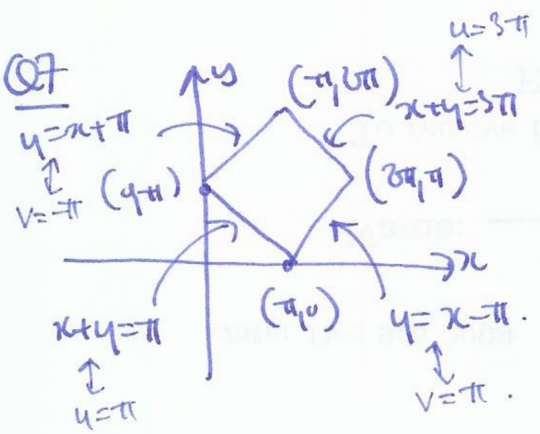
c)





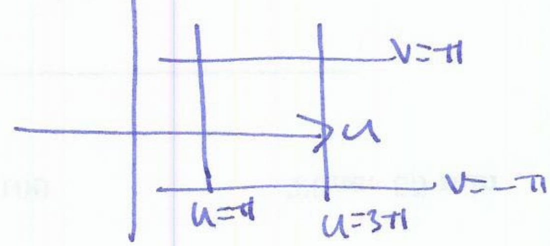
$$\int_0^{2\sqrt2}\int_0^{\frac12\sqrt{8-x^2}}\int_0^{8-x^2-2y^2} f(-)\,dz\,dy\,dx$$

Q7



$$T(u,v) = \left(\frac{u+v}{2}, \frac{u-v}{2}\right)$$

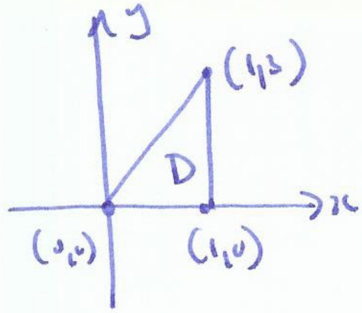
$x+y=u$
$x-y=v$



$$J = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix}\right| = |-1| = 1$$

$$\int_{\pi}^{3\pi}\int_{-\pi}^{\pi} v\sin(u)\,dv\,du = \int_{-\pi}^{\pi} v\,dv \underbrace{\int_{\pi}^{3\pi}\sin(u)\,du}_{0} = 0$$

Q8



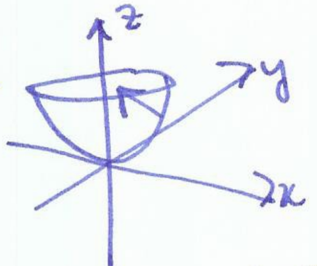
$$\int_{C=\partial D} \underline{F}\cdot d\underline{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\, dA$$

$$\underline{F} = \left\langle \sqrt{1+x^3},\ 2xy \right\rangle \quad \frac{\partial F_2}{\partial x} = 2y,\ \frac{\partial F_1}{\partial y} = 0$$

$$\int_0^1\int_0^{3x} 2y\, dy\, dx = \qquad \left[y^2\right]_0^{3x} = 9x^2$$

$$\int_0^1 9x^2\, dx = \left[3x^3\right]_0^1 = 3.$$

Q9



$$\iint_S \text{curl}(\underline{F})\, d\underline{S} = \int_{\partial S} \underline{F}\cdot d\underline{s} \qquad \underline{F} = \langle y^2, x, z^2\rangle$$

RHS: $c(\theta) = (\cos\theta, \sin\theta, 1)$ $\quad c'(\theta) = (-\sin\theta, \cos\theta, 0)$

$$\int_0^{2\pi} \langle \sin^2\theta, \cos\theta, 1\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle\, d\theta = \int_0^{2\pi} \underbrace{-\sin^3\theta}_{0} + \underbrace{\cos^2\theta}_{\pi}\, d\theta = \pi$$

LHS: $\cancel{T(u,v) = (u, v, u^2+v^2)}$ $\quad T(r,\theta) = (r\cos\theta, \sin\theta, r^2)$ $\quad 0 \le r \le 1,\ 0 \le \theta \le 2\pi$

$$\frac{\partial T}{\partial r} = (\cos\theta, \sin\theta, 2r) \qquad \underline{n} = (-2r^2\cos\theta, -2r^2\sin\theta, r)$$

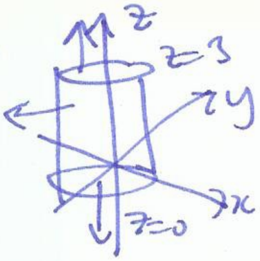
$$\frac{\partial T}{\partial \theta} = (-r\sin\theta, r\cos\theta, \cancel{2r}\ 0)$$

$$\text{curl}(\underline{F}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x & z^2 \end{vmatrix} = \langle 0, 0, 1-2y\rangle$$

$$\int_0^{2\pi}\int_0^1 \langle 0, 0, 1+4r^2\sin\theta\rangle \cdot \langle -2r^2\cos\theta, -2r^2\sin\theta, r\rangle\, dr\, d\theta.$$

$$= \int_0^{2\pi}\int_0^1 r + 4r^3 \underbrace{\sin\theta}_{0}\, dr\, d\theta = \pi, \text{ as required.}$$

Q10



$x^2+y^2 \le 1 \qquad \underline{F} = \langle x, y, -z \rangle.$

$\mathrm{div}(\underline{F}) = 1+1-1 = 1.$

$$\iiint 1 \, dV = \int_0^{2\pi}\int_0^1\int_0^3 r \, dz\, dr\, d\theta = 3\pi.$$

$$\iiint_W \mathrm{div}(\underline{F})\, dV = \iint_{\partial W} \underline{F}\cdot d\underline{S}.$$

bottom: $T(r,\theta) = (r\cos\theta, r\sin\theta, 0).\quad 0\le\theta\le 2\pi,\ 0\le r\le 1$

$$\left.\begin{aligned}\frac{\partial T}{\partial r} &= (\cos\theta, \sin\theta, 0)\\ \frac{\partial T}{\partial \theta} &= (-r\sin\theta, r\cos\theta, 0)\end{aligned}\right\}\ \underline{n} = (0, 0, r) \quad (\text{want } -\underline{n}).$$

$$\int_0^{2\pi}\int_0^1 \langle r\cos\theta, r\sin\theta, -0\rangle.\langle 0,0,r\rangle\, dr\, d\theta = \int_0^{2\pi}\int_0^1 0 \, dr\, d\theta = 0.$$

top: $T(r,\theta) = (r\cos\theta, r\sin\theta, 3) \quad \frac{\partial T}{\partial r}, \frac{\partial T}{\partial \theta}$ as above $\underline{n} = (0,0,r)$.

$$\int_0^{2\pi}\int_0^1 \langle r\cos\theta, r\sin\theta, -3\rangle.\langle 0,0,r\rangle\, dr\, d\theta = \int_0^{2\pi}\int_0^1 -3r \, dr\, d\theta = -3\pi$$

sides: $T(\theta,z) = (\cos\theta, \sin\theta, z) \quad 0\le z\le 3,\ 0\le\theta\le 2\pi$

$$\left.\begin{aligned}\frac{\partial T}{\partial \theta} &= (-\sin\theta, \cos\theta, 0)\\ \frac{\partial T}{\partial z} &= (0,0,1)\end{aligned}\right\}\ \underline{n} = (\cos\theta, \sin\theta, 0)$$

$$\int_0^3\int_0^{2\pi} \langle \cos\theta, \sin\theta, -z\rangle.\langle \cos\theta, \sin\theta, 0\rangle\, d\theta\, dz = \int_0^3\int_0^{2\pi} 1 \, d\theta\, dz = 6\pi$$

$6\pi - 3\pi = 3\pi$ as required.